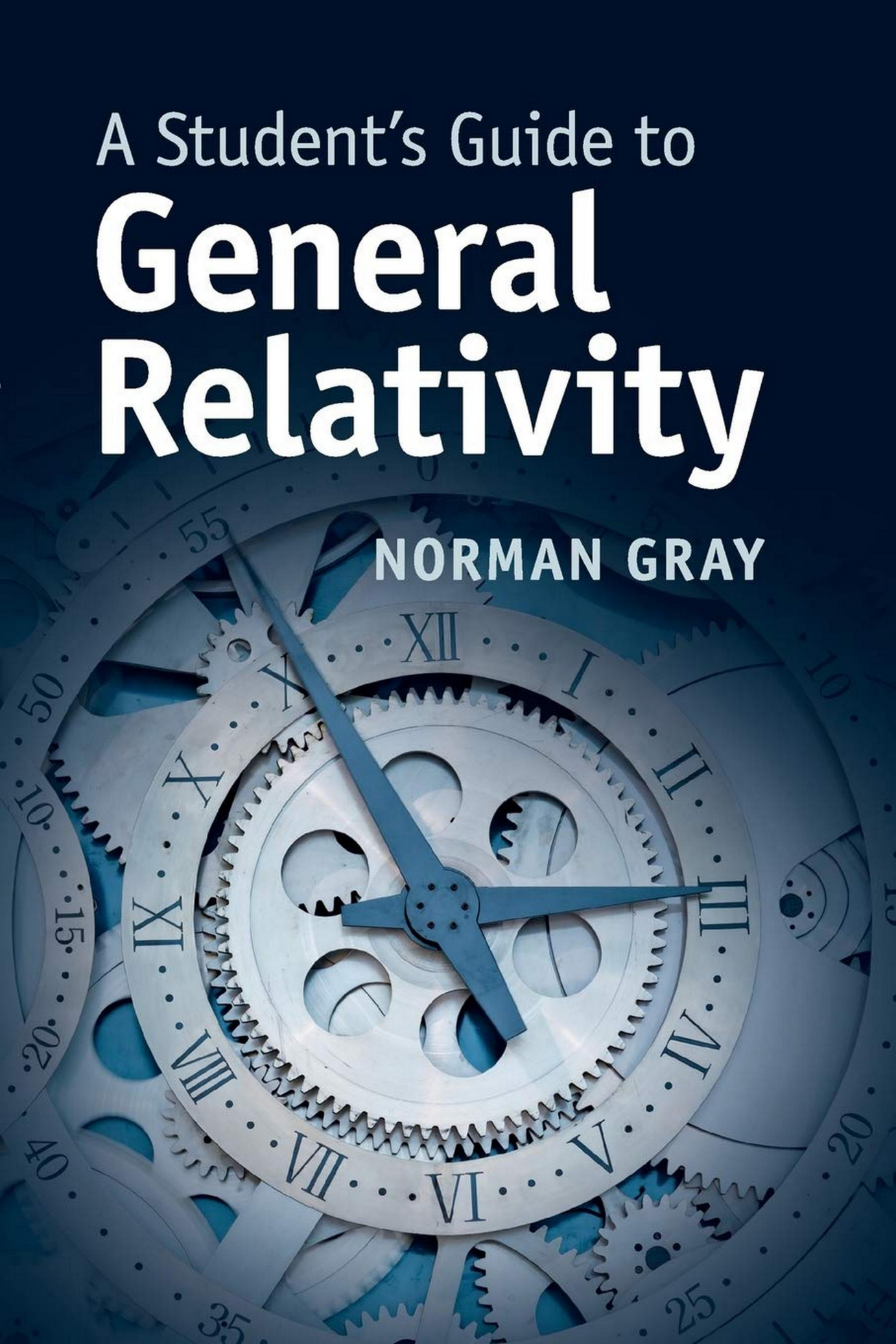
A Student's Guide to General Relativity

NORMAN GRAY



A Student's Guide to General Relativity

This compact guide presents the key features of General Relativity, to support and supplement the presentation in mainstream, more comprehensive undergraduate textbooks, or as a recap of essentials for graduate students pursuing more advanced studies. It helps students plot a careful path to understanding the core ideas and basic techniques of differential geometry, as applied to General Relativity, without overwhelming them. While the guide doesn't shy away from necessary technicalities, it emphasizes the essential simplicity of the main physical arguments. Presuming a familiarity with Special Relativity (with a brief account in an appendix), it describes how general covariance and the equivalence principle motivate Einstein's theory of gravitation. It then introduces differential geometry and the covariant derivative as the mathematical technology which allows us to understand Einstein's equations of General Relativity. The book is supported by numerous worked examples and exercises, and important applications of General Relativity are described in an appendix.

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Before thir eyes in sudden view appear
The secrets of the hoarie deep, a dark
Illimitable Ocean without bound,
Without dimension, where length, breadth, & highth,
And time and place are lost;
[. . .]
Into this wilde Abyss,
The Womb of nature and perhaps her Grave,
Of neither Sea, nor Shore, nor Air, nor Fire,
But all these in thir pregnant causes mixt
Confus'dly, and which thus must ever fight,
Unless th' Almighty Maker them ordain
His dark materials to create more Worlds,
Into this wild Abyss the warie fiend
Stood on the brink of Hell and look'd a while,
Pondering his Voyage: for no narrow frith
He had to cross.

John Milton, *Paradise Lost* , II, 890–920

But in the dynamic space of the living Rocket,
the double integral has a different meaning. To integrate
here is to operate on a rate of change so that time falls away:
change is stilled . . . 'Meters per second' will integrate
to 'meters.' The moving vehicle is frozen, in space,
to become architecture, and timeless.
It was never launched.
It will never fall.

Thomas Pynchon, *Gravity's Rainbow*

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Preface

This introduction to General Relativity (GR) is deliberately short, and is tightly focused on the goal of introducing differential geometry, then getting to Einstein's equations as briskly as possible.

There are four chapters:

Chapter 1 – Introduction and Motivation.
Chapter 2 – Vectors, Tensors, and Functions.
Chapter 3 – Manifolds, Vectors, and Differentiation.
Chapter 4 – Physics: Energy, Momentum, and Einstein's Equations.

The principal mathematical challenges are in Chapters 2 and 3, the first of which introduces new notations for possibly familiar ideas. In contrast, Chapters 1 and 4 represent the connection to physics, first as motivation, then as payoff. The main text of the book does not cover Special Relativity (SR), nor does it cover applications of GR to any significant extent. It is useful to mention SR, however, if only to fix notation, and it would be perverse to produce a book on GR without a mention of at least *some* interesting metrics, so both of these are discussed briefly in appendices.

When it comes down to it, there is not a huge volume of material that a physicist must learn before they gain a technically adequate grasp of Einstein's equations, and a long book can obscure this fact. We must learn how to describe coordinate systems for a rather general class of spaces, and then learn how to differentiate functions defined on those spaces. With that done, we are over the threshold of GR: we can define interesting functions such as the Energy-Momentum tensor, and use Einstein's equations to examine as many applications as we need, or have time for.

This book derives from a ten-lecture honours/masters course I have delivered for a number of years in the University of Glasgow. It was the first of a pair

of courses: this one was ‘the maths half’, which provided most of the maths required for its partner, which focused on various applications of Einstein’s equations to the study of gravity. The course was a compulsory one for most of its audience: with a smaller, self-selecting class, it might be possible to cover the material in less time, by compressing the middle chapters, or assigning readings; with a larger class and a more leisurely pace, we could happily spend a lot more time at the beginning and end, discussing the motivation and applications.

In adapting this course into a book, I have resisted the temptation to expand the text at each end. There are already many excellent but heavy tomes on GR – I discuss a few of them in Section 1.4.2 – and I think I would add little to the sum of world happiness by adding another. There are also shorter treatments, but they are typically highly mathematical ones, which don’t amuse everyone. Relativity, more than most topics, benefits from your reading multiple introductions, and I hope that this book, in combination with one or other of the mentioned texts, will form one of the building blocks in your eventual understanding of the subject.

As readers of any book like this will know, a lecture course has a *point*, which is either the exam at the end, or another course that depends on it. This book doesn’t have an exam, but in adapting it I have chosen to act as if it did: the book (minus appendices) has the same material as the course, in both selection and exclusion, and has the same practical goal, which is to lead the reader as straightforwardly as is feasible to a working understanding of the core mathematical machinery of GR. Graduate work in relativity will of course require mining of those heavier tomes, but I hope it will be easier to explore the territory after a first brisk march through it. The book is not designed to be dipped into, or selected from; it should be read straight through. Enjoy the journey.

Another feature of lecture courses and of Cambridge University Press’s *Student’s Guides*, which I have carried over to this book, is that they are bounded: they do not have to be complete, but can freely refer students to other texts, for details of supporting or corroborating interest. I have taken full advantage of this freedom here, and draw in particular on Schutz’s *A First Course in General Relativity* (2009), and to a somewhat lesser extent on Carroll’s *Spacetime and Geometry* (2004), aligning myself with Schutz’s approach except where I have a positive reason to explain things differently. This book is not a ‘companion’ to Schutz, and does not assume you have a copy, but it is deliberately highly compatible with it. I am greatly indebted both to these and to the other texts of Section 1.4.2.

In writing the text, I have consistently aimed for succinctness; I have generally aimed for one precise explanation rather than two discursive ones, while remembering that I am writing a physics text, and not a maths one. And in line with the intention to keep the destination firmly in mind, there are rather few major excursions from our route. The book is intended to be usable as a primary resource for students who need or wish to know some GR but who will not (yet) specialise in it, and as a secondary resource for students starting on more advanced material.

The text includes a number of exercises, and the density of these reflects the topics where my students had most difficulty. Indeed, many of the exercises, and much of the balance of the text, are directly derived from students' questions or puzzles. Solutions to these exercises can be downloaded at www.cambridge.org/gray.

Throughout the book, there are various passages, and a couple of complete sections, marked with 'dangerous bend' signs, like this one. They indicate supplementary details, material beyond the scope of the book which I think may be nonetheless interesting, or extra discussion of concepts or techniques that students have found confusing or misunderstandable in the past. If, again, this book had an exam, these passages would be firmly out of bounds.

Acknowledgements

These notes have benefitted from very thoughtful comments, criticism, and error checking, received from both colleagues and students, over the years this book's precursor course has been presented. The balance of time on different topics is in part a function of these students' comments and questions. Without downplaying many other contributions, Craig Stark, Liam Moore, and Holly Waller were helpfully relentless in finding ambiguities and errors.

The book would not exist without the patience and precision of Róisín Munnelly and Jared Wright of CUP. Some of the exercises and some of the motivation are taken, with thanks, from an earlier GR course also delivered at the University of Glasgow by Martin Hendry. I am also indebted to various colleagues for comments and encouragement of many types, in particular Richard Barrett, Graham Woan, Steve Draper, and Susan Stuart. For their precision and public-spiritedness in reporting errors, the author would like to thank Charles Michael Cruickshank, David Spaughton and Graham Woan.

1
Introduction

What is the problem that General Relativity (GR) is trying to solve? Section 1.1 introduces the principle of general covariance, the relativity principle, and the equivalence principle, which between them provide the physical underpinnings of Einstein's theory of gravitation.

We can examine some of these points a second time, at the risk of a little repetition, in Section 1.2, through a sequence of three thought experiments, which additionally bring out some immediate consequences of the ideas. It's rather a matter of taste, whether you regard the thought experiments as motivation for the principles, or as illustrations of them.

The remaining sections in this chapter are other prefatory remarks, about 'natural units' (in which the speed of light c and the gravitational constant G are both set to 1), and pointers to a selection of the many textbooks you may wish to consult for further details.

1.1 Three Principles

Newton's second law is

$$\mathbf{F} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}, \tag{1.1}$$

which has the special case, when the force $\mathbf{F}$ is zero, of $\mathrm{d}\mathbf{p}/\mathrm{d}t = 0$: The momentum is a conserved quantity in any force-free motion. We can take this as a statement of Newton's first law. In the standard example of first-year physics, of a puck moving across an ice rink or an idealised car moving along an idealised road, we can start to calculate with this by attaching a rectilinear coordinate system S to the rink or to the road, and discovering that

$$\mathbf{F} = m\mathbf{a} = m\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}, \tag{1.2}$$

from which we can deduce the constant-acceleration equations and, from that, all the fun and games of Applied Maths 1.

Alternatively, we could describe a coordinate system S' rotating about the origin of our rectilinear one with angular speed Ω, in which

$$\mathbf{F}' = m\mathbf{a}' = -m\Omega \times (\Omega \times \mathbf{r}') - 2m\Omega \times \frac{\mathrm{d}\mathbf{r}'}{\mathrm{d}t}, \tag{1.3}$$

and then derive the equations of constant acceleration from that. Doing so would not be wrong, but it would be perverse, because the underlying physical statement is the same in both cases, but the expression of it is more complicated in one frame than in the other. Put another way, Eq. (1.1) is physics, but the distinction between Eqs. (1.2) and (1.3) is merely mathematics.

This is a more profound statement than it may at first appear, and it can be dignified as

> *The principle of general covariance:* All physical laws must be invariant under all coordinate transformations.

A putative physical law that depends on the details of a particular frame – which is to say, a particular coordinate system – is one that depends on a mathematical detail that has no physical significance; we must rule it out of consideration as a physical law. Instead, Eq. (1.1) is a relation between two geometrical objects, namely a momentum vector and a force vector, and this illustrates the geometrical approach that we follow in this text: a physical law must depend only on geometrical objects, independent of the frame in which we realise them. In order to do calculations with it, we need to pick a particular frame, but that is incidental to the physical insight that the equation represents. The geometrical objects that we use to model physical quantities are vectors, one-forms, and tensors, which we learn about in Chapter 2.

It is necessary that the differentiation operation in Eq. (1.1) is also frame-independent. Right now, this may seem too obvious to be worth drawing attention to, but in fact a large part of the rest of this text is about defining differentiation in a way that satisfies this constraint. You may already have come across this puzzle, if you have studied the convective derivative in fluid mechanics or the tensor derivative in continuum mechanics, and you will have had hints of it in learning about the various forms of the Laplacian in different coordinate systems. See Section 1.3 for a preview.

It is also fairly obvious that Eq. (1.2) is a simpler expression than Eq. (1.3). This observation is not of merely aesthetic significance, but it prompts us to discover that there is a large class of frames where the expression of Newton's second law takes the same simple form as Eq. (1.2); these frames are the frames

that are moving with respect to S with a constant velocity $\mathbf{v}$, and we call each of the members of this class an *inertial frame* . In each inertial frame, motion is simple and, moreover, each inertial frame is related to another in a simple way: namely the *galilean transformation* in the case of pre-relativistic physics, and the *Lorentz transformation* in the case of Special Relativity (SR).

The fact that the observational effects of Newton's laws are the same in each inertial frame means that we cannot tell, from observation only of dynamical phenomena within the frame, which frame we are in. Put less abstractly, you can't tell whether you're moving or stationary, without looking outside the window and detecting movement relative to some other frame. Inertial frames thus have, or at least can be taken to have, a special status. This special status turns out, as a matter of observational fact, to be true not only of dynamical phenomena dependent on Newton's laws, but of all physical laws, and this also can be elevated to a principle.

> *The principle of relativity (RP):* (a) All true equations in physics (i.e., all 'laws of nature', and not only Newton's first law) assume the same mathematical form relative to all local inertial frames. Equivalently, (b) no experiment performed wholly within one local inertial frame can detect its motion relative to any other local inertial frame.

If we add to this principle the axiom that the speed of light is infinite, we deduce the galilean transformation; if we instead add the axiom that the speed of light is a frame-independent constant (an axiom that turns out to be amply confirmed by observation), we deduce the Lorentz transformation and Special Relativity. In SR, remember, we are obliged to talk of a four-dimensional coordinate frame, with one time and three space dimensions.

General Relativity – Einstein's theory of gravitation – adds further significance to the idea of the inertial frame. Here, an inertial frame is a frame in which SR applies, and thus the frame in which the laws of nature take their corresponding simple form. This definition, crucially, applies even in the presence of large masses where (in newtonian terms) we would expect to find a gravitational force. The frames thus picked out are those which are in *free fall*, either because they are in deep space far from any masses, or because they are (attached to something that is) moving under the influence of 'gravitation' alone. I put 'gravitation' in scare quotes because it is part of the point of GR to demote gravitation from its newtonian status as a distinct physical force to a status as a mathematical fiction – a conceptual convenience – which is no more real than centrifugal force.

The first step of that demotion is to observe that the force of gravitation (I'll omit the scare quotes from now on) is strangely independent of the

nature of the things that it acts upon. Imagine a frame sitting on the surface of the Earth, and in it a person, a bowl of petunias, and a radio, at some height above the ground: we discover that, when they are released, each of them will accelerate at the same rate towards the floor (Galileo is supposed to have demonstrated this same thing using the Tower of Pisa, careless of the health and safety of passers-by). Newton explains this by saying that the force of gravitation on each object is proportional to its gravitational mass (the gravitational 'charge', if you like); and the acceleration of each object, in response to that force, is proportional to its inertia, which is proportional to its inertial mass. Newton doesn't put it in those terms, of course, but he also fails to explain why the gravitational and inertial masses, which *a priori* have nothing to do with each other, turn out experimentally to be *exactly* proportional to each other, even though the person, the plant, the plantpot, and the radio broadcasting electromagnetic waves all exhibit very different physical properties.

Now imagine this same frame – or, for the sake of concreteness and the containment of a breathable atmosphere, a spacecraft – floating in space. Since spacecraft, observer, petunias, and radio are all equally floating in space, none will move with respect to another (or, if they are initially moving, they will continue to move with constant relative velocity). That is, Newton's laws work in their simple form in this frame, which we can therefore identify as an inertial frame.

If, now, we turn on the spacecraft's engines, then the spacecraft will accelerate, but the objects within it will not, until the spacecraft collides with them, and starts to accelerate them by pushing them with what we will at that point decide to call the cabin floor. Crucially – and, from this point of view, obviously – the sequence of events here is independent of the details of the structure of the ceramic plantpot, the biology of the observer and the petunias, and the electronic intricacies of the radio. If the spacecraft continues to accelerate at, say, $9.81\,\mathrm{m\,s^{-2}}$, then the objects now firmly on the cabin floor will experience a continuous force of one standard Earth gravity, and observers within the cabin will find it difficult to tell whether they are in an accelerating spacecraft or in a uniform gravitational field.

In fact we can make the stronger statement – and this is another physical statement which has been verified to considerable precision in, for example, the Eötvös experiments – that the observers will find it *impossible* to tell the difference between acceleration and uniform gravitation; and this is a third remark that we can elevate to a physical principle.

The Equivalence Principle (EP): Uniform gravitational fields are equivalent to frames that accelerate uniformly relative to inertial frames.

The EP is closely related to the observation that gravitational and inertial mass are strictly proportional; Rindler, for example, refers to this as the 'weak' equivalence principle (see Section 4.2.2).

We can summarise where we have got to as follows: (i) the principle of general covariance thus constrains the possible forms of statements of physical law, (ii) the EP and RP point to a privileged status of inertial frames in our search for further such laws, (iii) the RP gives us a link to the physics that we already know at this stage, and (iv) the EP gives us a link to the 'gravitational fields' that we want to learn more about.

These three principles make a variety of physical and mathematical points.

- The principle of general covariance restricts the category of mathematical statements that we are prepared to countenance as possible descriptions of nature. It says something about the relationship between physics and mathematics.
- The RP is either, in version (b) above, a straightforwardly physical statement or, in version (a), a physical statement in mathematical form. It picks out inertial frames as having a special status, and by saying that all inertial frames have equal status, it restricts the transformation between any pair of frames.
- The EP is also a physical statement. As we will examine further in Chapter 4, it further constrains the set of 'special' inertial frames, while retaining the idea that these inertial frames are physically indistinguishable, and exploring the constraints that that equivalence imposes.

By a 'physical statement' I mean a statement that picks out one of multiple mathematically consistent possibilities, and says that *this one* is the one that matches our universe. Mathematically, we *could* have a universe in which the galilean transformation works for all speeds, and the speed of light is infinite; but we don't.

Most of the statements in this section can be quibbled with, sometimes with great sophistication. The statement of the RP is quoted with minor adaptation from Barton (1999), who discusses the principle at book length in the context of SR. The wording of the EP is from Schutz (2009, §5.1), but Rindler (2006) discusses this with characteristic precision in his early chapters (distinguishing weak, strong, and semistrong variants of the EP), and Misner, Thorne and Wheeler (1973, §§7.2–7.3) discuss it with characteristic vividness. There is a minor industry devoted to the precise physical content of the EP and the principle of general covariance, and to their logical relationship to Einstein's theory of gravity. This industry is discussed at substantial length by Norton (1993), and subsequent texts quoting it, but it

does not seem to contribute usefully to an elementary discussion such as this one, and I have thought it best to keep the account in this section as compact and as straightforward as possible, while noting that there is much more one can go on to think about.

1.2 Some Thought Experiments on Gravitation

At the risk of some repetition, we can make the same points again, and make some further interesting deductions, through a sequence of thought experiments.

1.2.1 The Falling Lift

Recall from SR that we may define an inertial frame to be one in which Newton's laws hold, so that particles that are not acted on by an external force move in straight lines at a constant velocity. In Misner, Thorne, and Wheeler's words, inertial frames and their time coordinates are defined so that motion looks simple. This is also the case if we are in a box far away from any gravitational forces, we may identify that as a *local inertial frame* (we will see the significance of the word 'local' later in the chapter). Another way of removing gravitational forces – less extreme than going into deep space – is to put ourselves in free fall. Einstein asserted that these two situations are indeed fully equivalent, and defined an inertial frame as one in free fall.

Objects at rest in an inertial frame – in either of the equivalent situations of being far away from gravitating matter or freely falling in a gravitational field – will stay at rest. If we accelerate the box-cum-inertial-frame, perhaps by attaching rockets to its 'floor', then the box will accelerate but its contents won't; they will therefore move towards the floor at an increasing speed, from the point of view of someone in the box.[1] This will happen irrespective of the mass or composition of the objects in the box; they will all appear to increase their speed at the same rate.

Note that I am carefully *not* using the word 'accelerate' for the change in speed of the objects in the box with respect to that frame. We reserve that word for the physical phenomenon measured by an accelerometer, and the result of a real force, and try to avoid using it (not, I fear, always successfully) to refer

[1] By 'point of view' I mean 'as measured with respect to a reference frame fixed to the box', but such circumlocution can distract from the point that this is an *observation* we're talking about – we can see this happening.

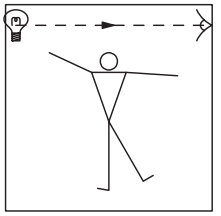

Figure 1.1 A floating box.

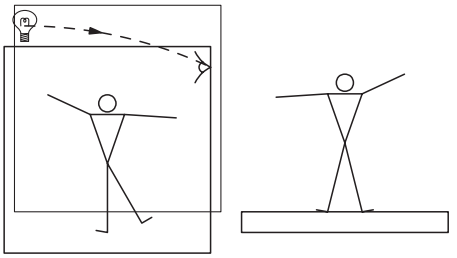

Figure 1.2 A free-fall box.

to the second derivative of a position. Depending on the coordinate system, the one does not always imply the other, as we shall see later.

This is very similar to Galileo's observation that all objects fall under gravity at the same rate, irrespective of their mass or composition. Einstein supposed that this was not a coincidence, and that there was a deep equivalence between acceleration and gravity (we shall see later, in Chapter 4, that the force of gravity that we feel while standing in one place is the result of us being accelerated away from the path we would have if we were in free fall). He raised this to the status of a postulate: the Equivalence Principle.

Imagine being in a box floating freely in space, and imagine shining a torch horizontally across it from one wall to the other (Figure 1.1). Where will the beam end up? Obviously, it will end up at a point on the wall directly opposite the torch. There's nothing exotic about this. The EP tells us that the same must happen for a box in free fall. That is, a person inside a falling lift would observe the torch beam to end up level with the point at which it was emitted, in the (inertial) frame of the lift. This is a straightforward and unsurprising use of the EP. How would this appear to someone watching the lift fall?

Since the light takes a finite time to cross the lift cabin, the spot on the wall where it strikes will have dropped some finite (though small) distance, and so will be lower than the point of emission, in the frame of someone watching this from a position of safety (Figure 1.2). That is, this non-free-fall observer will measure the light's path as being curved in the gravitational field. Even massless light is affected by gravity. [Exercise 1.1]

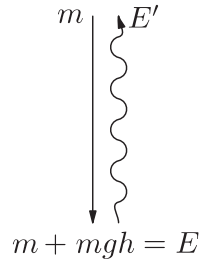

Figure 1.3 The Pound-Rebka experiment.

1.2.2 Gravitational Redshift

Imagine dropping a particle of mass m through a distance h. The particle starts off with energy m ($E = mc^2$, with $c = 1$; see Section 1.4.1), and ends up with energy $E = m + mgh$ (see Figure 1.3). Now imagine converting all of this energy into a single photon of energy E, and sending it up towards the original position. It reaches there with energy E', which we convert *back* into a particle.[2] Now, either we have invented a perpetual motion machine, or else $E' = m$:

$$E' = m = \frac{E}{1 + gh}, \tag{1.4}$$

and we discover that a photon loses energy as a necessary consequence of climbing through a gravitational field, and as a consequence of our demand that energy be conserved.

This energy loss is termed *gravitational redshift*, and it (or rather, something very like it) has been confirmed experimentally, in the 'Pound-Rebka experiment'. It's also sometimes referred to as 'gravitational doppler shift', but inaccurately, since it is not a consequence of relative motion, and so has nothing to do with the doppler shift that you are familiar with.

Light, it seems, can tell us about the gravitational field it moves through.

1.2.3 Schild's Photons

Imagine firing a photon, of frequency f, from an event A to an event B spatially located directly above it in a gravitational field (see Figure 1.4). As we discovered in the previous section, the photon will be redshifted to a new frequency f'. After some number of periods n, we repeat this, and send up another photon (between the points marked A' and B' on the space-time diagram).

[2] As described, this is kinematically impossible, since we cannot do this and conserve momentum, but we can imagine sending distinct particles back and forth, conserving just energy; this would have an equivalent effect, but be more intricate to describe precisely.

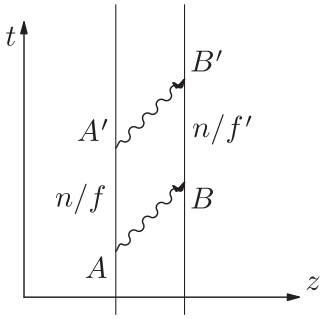

Figure 1.4 Schild's photons.

Photons are a kind of clock, in that the interval between 'wavecrests', $1/f$, forms a kind of 'tick'. The length of this tick will be measured to have different numerical values in different frames, but the start and end of the interval nonetheless constitute two frame-independent events.

Presuming that the source and receiver are not in relative motion, the intervals AB and $A'B'$ will be the same (I've drawn these as straight lines on the diagram, but the argument doesn't depend on that). However, the intervals AA' and BB' comprise the same number n of periods, which means that the intervals in time, n/f and n/f', as measured by local clocks, are *different* . That is, we have not constructed the parallelogram we might have expected, and have therefore discovered that the geometry of this space-time is not flat geometry we might have expected, and that this is purely as a result of the presence of the gravitational field through which we are sending the photons.

Finding out more about this geometry is what we aim to do in this text.

The 'Schild's photons' argument, and a version of the gravitational redshift argument, first appeared in Schild (1962), where both are presented in careful and precise detail. The subtleties are important, but the arguments in the sections earlier in this chapter, though slightly schematic, contain the essential intuition. Schild's paper also includes a thoughtful discussion of what parts of GR are and are not addressed by experiment.

1.2.4 Tides and Geodesic Deviation (and Local Frames)

Consider two particles, A and B, both falling towards the earth, with their height from the centre of the earth given by $z(t)$ (Figure 1.5). They start off level with each other and separated by a horizontal distance $\xi(t)$.

From the diagram, the separation $\xi(t)$ is proportional to $z(t)$, so that $\xi(t) = kz(t)$, for some constant k. The gravitational force on a particle of mass m at altitude z is $F = GMm/z^2$, thus

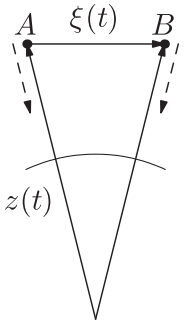

Figure 1.5 Two falling particles.

$$\frac{\mathrm{d}^2\xi}{\mathrm{d}t^2} = k\frac{\mathrm{d}^2 z}{\mathrm{d}t^2} = -k\frac{F}{m} = -k\frac{GM}{z^2} = -\xi\frac{GM}{z^3}.$$

This tells us that the inertial frames attached to these freely falling particles approach each other at an increasing speed (that is, they 'accelerate' towards each other in the sense that the second derivative of their separation is non-zero, but since they are in free fall, there is no physical acceleration that an observer in the frame would feel as a push).

If A and B are two observers in inertial frames (or inertial spacecraft), then we have said that they cannot distinguish between being in space far from any gravitating masses, and being in free fall near a large mass. If instead they found themselves at opposite ends of a giant free-falling spacecraft, then they would find themselves drifting closer to each other as the spacecraft fell, in apparent violation of Newton's laws. Is there a contradiction here?

No. The EP as quoted in Section 1.1 talked of *uniform* gravitational fields, which this is not. Also, both the RP of that section, and the discussion in Section 1.2.1, talked of *local* inertial frames. A lot of SR depends on inertial frames having infinite extent: if I am an inertial observer, then any other inertial observer must be moving at a constant velocity with respect to me. In GR, in contrast, an inertial frame is a local approximation (indeed it is fully accurate only at a point, an important issue we will return to later), and if your measurement or experiment is sufficiently extended in space or time, or if your instruments are sufficiently accurate, then you will be able to detect tidal forces in the way that A and B have done in this thought experiment.

If A and B are plummeting down lift shafts, in free fall, on opposite sides of the earth, then they are inertial observers, but they are 'accelerating' with respect to one another. This means that, if I am one of these inertial observers, then (presuming I do not have more pressing things to worry about) I cannot use SR to calculate what the other inertial observer would measure in their frame, nor calculate what I would measure if I observed a bit of physics that I understand, which is happening in the other inertial observer's frame.

But this is precisely what I do want to do, supposing that the bit of physics in question is happening in free fall in the accretion disk surrounding a black hole, and I want to interpret what I am seeing through my telescope. Gravitational redshift of spectral lines is just the beginning of it.

It is GR that tells us how we must patch together such disparate inertial frames. [Exercise 1.2]

1.3 Covariant Differentiation

Like many other parts of physics, the study of gravitation depends on differential equations, and working with differential equations depends (obviously) on being able to differentiate. A large fraction of this book – essentially all of Chapter 3 – is taken up with learning how to define differentiation in a curved space-time.

In many ways the key section of the book is Section 3.3.2. That section builds directly on the definition of differentiation that you learned about in school. For some function $f : \mathbb{R} \to \mathbb{R}$,

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}.$$

That definition is straightforward because it's obvious what $f(x + h) - f(x)$ means, and it's obvious how we divide that by a number.

In a curved space-time, however, a naive approach won't work, because the objects we want to differentiate are vectors (or other geometrical objects such as tensors, which we will learn about next, in Chapter 2), and the way we subtract them must be independent of any particular choice of coordinate system. We can see part of the problem even in two dimensions: while it is easy to see how to subtract two cartesian vectors (we simply work component by component), it is less clear how to subtract two vectors expressed in polar coordinates. If we go on to think about how to define and perform arithmetical operations on vectors defined on the surface of a sphere – a two-dimensional surface with intrinsic curvature – things become yet more subtle (think of the difference between plane and spherical trigonometry). All that said, the intuition that lies behind the definition earlier in this section is the same intuition that underlies the more elaborate maths of Chapter 3. Hold on to that thought.

In the next two chapters we will approach these problems step by step, and return to physics in Chapter 4, when we get a chance to apply these ideas in developing Einstein's equations for the structure of space-time. Appendix B is all about further application of the tools we develop in this one. The sequence of ideas is shown in Figure 1.6.

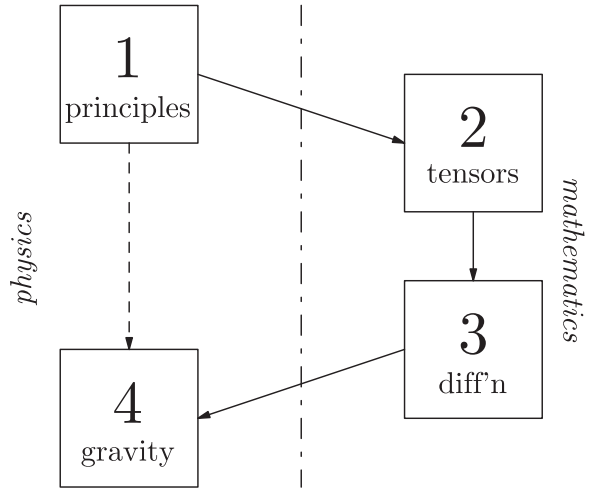

Figure 1.6 The sequence of ideas. In Chapters 2 and 3 we examine the mathematical technology that we will need to turn the principles of Chapter 1 into the physics of Chapter 4.

1.4 A Few Further Remarks

1.4.1 Natural Units

In SR, we normally use *natural units* (also *geometrical units*, and not quite the same thing as *Planck units*), in which we use the same units, metres, to measure both distance and time, with the result that we measure distance in these two directions in space-time using the same units (because of the high speed of light, metres and seconds are otherwise absurdly mismatched). We extend this in GR, but now measuring mass in metres also. First, a recap of natural units in SR.

It is straightforward to measure distances in time-units, and we do this naturally when we talk of Edinburgh being 50 minutes from Glasgow (maintenance works permitting), or the earth being 8 light-minutes from the sun, or the nearest star being a little more than 4 light years away. In fact, since 1981 or so, the International Standard definition of the metre is that it is the distance light travels in 1/299,792,458 seconds; that is, the speed of light is precisely $c = 299{,}792{,}458\,\mathrm{m\,s^{-1}}$ by definition, and so c is therefore demoted to being merely a conversion factor between two different units of distance, namely the metre and the (light-)second.

Alternatively, we can decide that this relation gives us permission to think of the metre as a (very small) unit of time: specifically the time it takes for light to travel a distance of one metre (about 3.3 nanoseconds-of-time).

There are several advantages to this: (i) In relativity, space and time are not really distinct, and having different units for the two 'directions' can obscure this; (ii) In these units, light travels a distance of one metre in a time of one metre, giving the speed of light as an easy-to-remember, and dimensionless, $c = 1$; (iii) If we measure time in metres, then we no longer need the conversion factor c in our equations, which are consequently simpler. We also quote other speeds in these units of metres per metre, so that all speeds are dimensionless and less than one.

Of these three points, the first is by far the most important.

Writing $c = 1 = 3 \times 10^8\,\mathrm{m\,s^{-1}}$ (dimensionless) looks rather odd, until we read 'seconds' as units of length. In the same sense, the inch is defined to be precisely 25.4 mm long, and this figure of 25.4 is merely a conversion factor between two different, and only historically distinct, units of length. We write this as $1\,\mathrm{in} = 25.4\,\mathrm{mm}$ or, equivalently but unconventionally, as $1 = 25.4\,\mathrm{mm\,in^{-1}}$.

Consider converting $10\,\mathrm{J} = 10\,\mathrm{kg\,m^2\,s^{-2}}$ to natural units. Since $c = 1$, we have $1\,\mathrm{s} = 3 \times 10^8\,\mathrm{m}$, and so $1\,\mathrm{s^{-2}} = (9 \times 10^{16})^{-1}\,\mathrm{m^{-2}}$. So $10\,\mathrm{kg\,m^2\,s^{-2}} = 10\,\mathrm{kg\,m^2} \times (9 \times 10^{16})^{-1}\,\mathrm{m^{-2}} = 1.1 \times 10^{-16}\,\mathrm{kg}$. Recalling SR's $E = \gamma mc^2 = \gamma m$, it should be unsurprising that, in the 'right' units, mass has the same units as other forms of energy.

In GR it is also usual to use units in which the gravitational constant is $G = 1$. That means that the expression $1 = G = 6.673 \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}} = 7.414 \times 10^{-28}\,\mathrm{m\,kg^{-1}}$ becomes a conversion factor between kilogrammes and the other units. This, for example, gives the mass of the sun, in these units, as $M_\odot \approx 1.5\,\mathrm{km}$.

It is easy, once you have a little practice, to convert values and equations between the different systems of units. Throughout the rest of this book, I will quote equations in units where $c = 1$, and, when we come to that, $G = 1$, so that the factors c and G disappear from the equations. [Exercise 1.3]

1.4.2 Further Reading

When learning relativity, even more than with other subjects, you benefit from reading things multiple times, from different authors, and from different points of view. I mention a couple of good introductions here, but there is really no substitute for going to the appropriate section in a library, looking through the books there, and finding one that makes sense to *you*.

I presume you are familiar with SR. There is a brief summary of SR in Appendix A, which is intended to be compatible with this book.

This book is significantly aligned with Schutz (2009) (hereafter simply 'Schutz'), in the sense that this is the book closest in style to this text; also, I will occasionally direct you to particular sections of it, for details or proofs.

Other textbooks you might want to look at follow. You may want to use these other books to take your study of the subject further. But you might also use them (perhaps a little cautiously) to test your understanding as you go along, by comparing what you have read here with another author's approach.

- Carroll (2004) is very good. Although it's mathematically similar, the order of the material, and the things it stresses, are sufficiently different from this book and Schutz that it might be confusing. However, that difference is also a major virtue: the book introduces topics clearly, and in a way that usefully contrasts with my way. Also, Carroll's relativity lecture notes from a few years ago, which are a precursor of the book, are easily findable on the Internet.
- Rindler (2006) always explains the physics clearly, distinguishing successively strong variants of the EP, and the motivation for GR (his first two chapters are, incidentally, notably excellent in their careful explanation of the conceptual basis of SR). However Rindler is now rather old-fashioned in many respects, in particular in its treatment of differential geometry, which it introduces from the point of view of coordinate transformations, rather than the geometrical approach we use later in the book. Earlier editions of this book are equally valuable for their insight.
- Similarly, again, Narlikar (2010) is worthwhile looking at, to see if it suits you. The mathematical approach is one which introduces vectors and tensors via components (like Rindler), rather than the more functional approach we'll use here. Narlikar is good at transmitting mathematical and physical insights.
- Misner, Thorne, and Wheeler (1973) is a glorious, comprehensive, doorstop of a book. Its distinctive prose style and typographical oddities have fans and detractors in roughly equal numbers. Chapter 1 in particular is worth reading for an overview of the subject. MTW is, incidentally, highly compatible in style with the introduction to SR found in Taylor and Wheeler's excellent *Spacetime Physics* (1992).
- Wald (1984) is comprehensive and has long been a standby of undergraduate- and graduate-level GR courses.
- Hartle (2003) is more recent and similarly popular, with a practical focus.

This is a pretty mathematical topic, but it is supposed to be a *physics* book, so we're looking for the physical insights, which can easily become buried beneath the maths.

- Another Schutz book, *Gravity from the Ground Up* Schutz (2003), aims to cover all of gravitational physics from falling apples to black holes using the minimum of maths. It won't help with the differential geometry, but it'll supply lots of insight.
- Longair (2003) is excellent. The section on GR (only a smallish part of the book) is concerned with motivating the subject rather than doing a lot of maths, and is in a seat-of-the-pants style that might be to your taste.

There are also many more advanced texts. The following are graduate-level texts, and so reach well beyond the level of this book. They are mathematically very sophisticated. If, however, your tastes and experience run that way, then the introductory chapters of these books might be instructive, and give you a taste of the vast wonderland of beautiful maths that can be found in this subject. They can also be useful as a way of compactly summarising material you have come to understand by a more indirect route.

- Chapter 1 of Stewart (1991) covers more than the content of this course in its first 60 laconic pages.
- *Geometrical Methods of Mathematical Physics* (Schutz, 1980) is a delightful book, which explains the differential geometry clearly and sparsely, including applications beyond relativity and cosmology. However, it appeals only to those with a strong mathematical background; it may cause alarm and despondency in others.
- Hawking and Ellis (1973), chapter 2, covers more than all the differential geometry of this book.

1.4.3 Notation Conventions, Here and Elsewhere

I use overlines to denote vectors: $\overline{A}$. This is consistent with Schutz (1980), but relatively rare elsewhere; it seems neater to me than the over-arrow version $\vec{A}$ (as well as easier to write by hand). One-forms are denoted with a tilde: $\widetilde{p}$. Tensors are in sans serif: g. For a summary of other notation conventions, see Appendix C.

There are a number of different *sign conventions* in use in relativity books. The conventions used in this book match those in Schutz, MTW (1973), Schutz (1980), and Hawking and Ellis (1973). We can summarise the conventions for

Table 1.1 Sign conventions in various texts. This text also matches Hawking and Ellis (1973) and MTW. References are to equation numbers in the corresponding texts, except where indicated. For explanations, see Eq. (1.5).

	Riemann	Ricci	Einstein	metric
This text	+ (3.49)	+ (4.16)	+ (4.37)	+ (2.33)
Schutz	+ (6.63)	+ (6.91)	+ (8.7)	+ (3.1)
Carroll	+ (3.113)	+ (3.144)	+ (4.44)	+ (1.15)
Rindler	+ (8.20)	− (8.31)	− (9.71)	− (7.12)
Stewart	− (1.9.4)	+ (1.9.12)	− (1.13.5)	− (§1.10)

a few texts (imitating MTW's corresponding table) in Table 1.1. In this table, the signs are

$$\begin{aligned} \pm R^{i}{}_{jkl} &= \Gamma^{i}_{jl,k} - \Gamma^{i}_{jk,l} + \Gamma^{i}_{\sigma k}\Gamma^{\sigma}_{jl} - \Gamma^{i}_{\sigma l}\Gamma^{\sigma}_{jk} \\ \pm R_{\beta\nu} &= R^{\mu}{}_{\beta\mu\nu} \\ G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu} &= \pm 8\pi T_{\mu\nu} \\ \pm \eta_{\mu\nu} &= \mathrm{diag}(-1, +1, +1, +1) \end{aligned} \tag{1.5}$$

Exercises

Here and in the following chapters, the notations d^{+}, d^{-}, u^{+}, and so on, indicate questions that are slightly more or less difficult, or more useful, than others.

Exercise 1.1 (§ 1.2.1) A photon is sent across a box of width h sitting in space, while it is being accelerated at $1g$, in the same direction, by a rocket. What is the frequency (or energy) of the photon when it is absorbed by a detector on the other side of the box? Use the Doppler redshift formula $\nu_{\mathrm{em}}/\nu_{\mathrm{obs}} = 1 + v$ (in units where $c = 1$), and note that the box will not move far in this time. How does this link to other remarks in this section? $[u^{+}]$

Exercise 1.2 (§ 1.2.4) If two 1 kg balls, 1m apart, fall down a lift shaft near the surface of the earth, how much is their tidal acceleration towards each other? How much is their acceleration towards each other as a result of their mutual gravitational attraction?

Exercise 1.3 (§ 1.4.1) Convert the following to units in which $c = 1$: (a) 10 J; (b) lightbulb power, 100 W; (c) Planck's constant, $h = 1.05 \times 10^{-34}$ J s;

(d) velocity of a car, $v = 30\,\mathrm{m\,s^{-1}}$; (e) momentum of a car, $3 \times 10^4\,\mathrm{kg\,m\,s^{-1}}$; (f) pressure of 1 atmosphere, $10^5\,\mathrm{N\,m^{-2}}$; (g) density of water, $10^3\,\mathrm{kg\,m^{-3}}$; (h) luminosity flux, $10^6\,\mathrm{J\,s^{-1}\,cm^{-2}}$.

Convert the following to physical units (SI): (i) velocity, $v = 10^{-2}$; (j) pressure $10^{19}\,\mathrm{kg\,m^{-3}}$; (k) time $10^{18}\,\mathrm{m}$; (l) energy density $u = 1\,\mathrm{kg\,m^{-3}}$; (m) acceleration $10\,\mathrm{m^{-1}}$; (n) the Lorentz transformation, $t' = \gamma(t - vx)$; (o) the 'mass-shell' equation $E^2 = p^2 + m^2$. Problem slightly adapted from Schutz (2009, ch.1).

2

Vectors, Tensors, and Functions

At this point we take a holiday from the physics, in favour of mathematical preliminaries. This chapter is concerned with defining vectors, tensors, and functions reasonably carefully, and showing how they are linked with the notion of coordinate systems. This will take us to the point where, in Chapter 3, we can talk about doing calculus with these objects. You may well be familiar with many of the mathematical concepts in this chapter – functions, vector spaces, vector bases, and basis transformations – but I will (re)introduce them in this chapter with a slightly more sophisticated mathematical notation, which will allow us to make use of them later. The exception to that is tensors, which may have seemed slightly gratuitous, if you have encountered them at all before; they are vital in relativity.

2.1 Linear Algebra

The material in this section will probably be, if not familiar, at least recognisable to you, though possibly with new notation. After this section, I'm going to assume you are comfortable with both the concepts and the notation; you may wish to recap some of your first- or second-year maths notes. See also Schutz's appendix A.

Here, and elsewhere in this book, the idea of *linearity* is of crucial importance; it is not, however, a complicated notion. Consider a function (or operator or other object) f, objects $\mathbf{x}$ and $\mathbf{y}$ in the domain of f, and numbers $\{a, b\} \in \mathbb{R}$: if $f(a\mathbf{x} + b\mathbf{y}) = af(\mathbf{x}) + bf(\mathbf{y})$, then the function f is said to be *linear*. Thus the function $f = ax$ is linear in x, but $f = ax + b$, $f = ax^2$ and $f = \sin x$ are not; matrix multiplication is linear in the (matrix) arguments, but the rotation of a solid sphere (for example) is not linear in the Euler angles (note that although you might refer to $f(x) = ax + b$ as a 'straight

line graph', or might refer to it as linear in other contexts, in this formal sense it is not a linear function, because $f(2x) \neq 2f(x)$).

2.1.1 Vector Spaces

Mathematicians use the term 'vector space' to refer to a larger set of objects than the pointy things that may spring to your mind at first.

A set of objects V is called a *vector space* if it satisfies the following axioms (for $A, B \in V$ and $a \in \mathbb{R}$):

1. Closure: there is a symmetric binary operator '+', such that $A + B = B + A \in V$.
2. Identity: there exists an element $0 \in V$, such that $A + 0 = A$.
3. Inverse: for every $A \in V$, there exists an element $B \in V$ such that $A + B = 0$ (incidentally, these first three properties together mean that V is classified as an *abelian group*).
4. Multiplication by reals: for all a and all A, $aA \in V$ and $1A = A$.
5. Distributive: $a(A + B) = aA + aB$.

The obvious example of a vector space is the set of vectors that you learned about in school, but crucially, *anything* that satisfies these axioms is also a vector space.

Vectors $A_1, \ldots, A_n$ are *linearly independent* (LI) if $a_1A_1 + a_2A_2 + \cdots + a_nA_n = 0$ implies $a_i = 0, \forall i$. The *dimension* of a vector space, n, is the largest number of LI vectors that can be found. A set of n LI vectors A_i in an n-dimensional space is said to *span* the space, and is termed a *basis* for the space. Then is it a theorem that, for every vector $B \in V$, there exists a set of numbers $\{b_i\}$ such that $B = \sum_{i=1}^{n} b_iA_i$; these numbers $\{b_i\}$ are the *components* of the vector B with respect to the basis $\{A_i\}$.

One can (but need not) define an *inner product* on a vector space: the inner product between two vectors A and B is written $A \cdot B$ (yes, the dot-product that you know about is indeed an example of an inner product; also note that the inner product is sometimes written $\langle A, B \rangle$, but we will reserve that notation, here, to the contraction between a vector and a one-form, defined in Section 2.2.1). This is a symmetric, linear, operator that maps pairs of vectors to the real line. That is (i) $A \cdot B = B \cdot A$, and (ii) $(aA + bB) \cdot C = aA \cdot C + bB \cdot C$. Two vectors, A and B, are *orthogonal* if $A \cdot B = 0$. An inner-product is *positive-definite* if $A \cdot A > 0$ for all $A \neq 0$, or indefinite otherwise. The *norm* of a vector A is $|A| = |A \cdot A|^{1/2}$. The symbol δ_{ij} is the *Kronecker delta symbol*, defined as

$$\delta_{ij} \equiv \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \tag{2.1}$$

(throughout the book, we will use variants of this symbol with indexes raised or lowered – they mean the same: $\delta_{ij} = \delta^i{}_j = \delta_i{}^j$; see the remarks about this object at the end of Section 2.2.6). A set of vectors $\{e_i\}$ such that $e_i \cdot e_j = \delta_{ij}$ (that is, all orthogonal and with unit norm) is an *orthonormal* basis. It is a theorem that, if $\{b_i\}$ are the components of an arbitrary vector B in this basis, then $b_i = B \cdot e_i$. [Exercises 2.1 and 2.2]

2.1.2 Matrix Algebra

An $m \times n$ matrix A is a mathematical object that can can be represented by a set of *elements* denoted A_{ij}, via

$$\mathsf{A} = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{pmatrix}.$$

You know how to define addition of two $m \times n$ matrices, and multiplication of a matrix by a scalar, and that the result in both cases is another matrix, so the set of $m \times n$ matrices is another example of a vector space. You also know how to define matrix multiplication: a vector space with multiplication defined is an *algebra*, so what we are now discussing is matrix algebra.

A square matrix (that is, $n \times n$) may have an inverse, written A^{-1}, such that $\mathsf{A}\mathsf{A}^{-1} = \mathsf{A}^{-1}\mathsf{A} = 1$ (one can define left- and right-inverses of non-square matrices, but they will not concern us). The unit matrix $\mathbb{1}$ has elements δ_{ij}.

You can define the *trace* of a square matrix, as the sum of the diagonal elements, and define the *determinant* by the usual intricate formula. Since both of these are invariant under a similarity transformation ($A \mapsto P^{-1}AP$), the determinant and trace are also the product and sum, respectively, of the matrix's eigenvalues.

Make sure that you are in fact familiar with the matrix concepts in this section.

2.2 Tensors, Vectors, and One-Forms

Most of the rest of this book is going to be talking about tensors one way or another, so we had better grow to love them now. See Schutz, chapter 3.

I am going to introduce tensors in a rather abstract way here, in order to emphasise that they are in fact rather simple objects. Tensors will become a little more concrete when we introduce tensor components shortly. and in the rest of the book we will use these extensively, but introducing components from the outset can hide the geometrical primitiveness of the underlying objects. I provide some specific examples of tensors in Section 2.2.2.

2.2.1 Definition of Tensors

For each $M, N = 0, 1, 2, \ldots$, the set of $\binom{M}{N}$ tensors is a set that obeys the axioms of a vector space from Section 2.1.1. Three of these sets of tensors have special names: a $\binom{0}{0}$ tensor is just a scalar function that maps $\mathbb{R} \to \mathbb{R}$; we refer to a $\binom{1}{0}$ tensor as a *vector* and write it as $\overline{A}$, and refer to a $\binom{0}{1}$ tensor as a *one-form* , written $\widetilde{A}$. The clash with the terminology of Section 2.1.1 is unfortunate (because *all* of these objects are 'vectors' in the terminology of that section), but from now on when we refer to a 'vector space', we are referring to Section 2.1.1, and when we refer to 'vectors', we are referring specifically to $\binom{1}{0}$ tensors.

For the moment, you can perfectly reasonably think of vectors as exactly the type of vectors you are used to – a thing with a magnitude and a direction in space. In Chapter 3, we will introduce a new definition of vectors which is of crucial importance in our development of GR.

Definition: A $\binom{M}{N}$ *tensor* is a function, linear in each argument, which takes M one-forms and N vectors as arguments, and maps them to a real number.

Because we said that an $\binom{M}{N}$ tensor was an element of a vector space, we already know that if we add two $\binom{M}{N}$ tensors, or if we multiply an $\binom{M}{N}$ tensor by a scalar, then we get another $\binom{M}{N}$ tensor. This definition does seem very abstract, but most of the properties we are about to deduce follow directly from it.

For example, we can write the $\binom{2}{1}$ tensor T as

$$\mathsf{T}(\widetilde{\cdot}, \widetilde{\cdot}, \overline{\cdot}),$$

to emphasise that the function has two 'slots' for one-forms and one 'slot' for a vector. When we insert one-forms $\widetilde{p}$ and $\widetilde{q}$, and vector $\overline{A}$, we get $\mathsf{T}(\widetilde{p}, \widetilde{q}, \overline{A})$, which, by our definition of a tensor, we see must be a pure number, in $\mathbb{R}$. Note that this 'dots' notation is an informal one, and though I have chosen to write this in the following discussion with one-form arguments all to the left of vector ones, this is just for the sake of clarity: in general, the $\binom{1}{1}$ tensor $\mathsf{T}(\overline{\cdot}, \widetilde{\cdot})$ is a perfectly good tensor, and distinct from the $\binom{1}{1}$ tensor $\mathsf{T}(\widetilde{\cdot}, \overline{\cdot})$.

Note firstly that there is nothing in the definition of a tensor that states that the arguments are interchangeable, thus, in the case of a $\binom{0}{2}$ tensor $\mathsf{U}(\overline{\cdot}, \overline{\cdot})$, $\mathsf{U}(\overline{A}, \overline{B}) \neq \mathsf{U}(\overline{B}, \overline{A})$ in general: if in fact $\mathsf{U}(\overline{A}, \overline{B}) = \mathsf{U}(\overline{B}, \overline{A}), \forall \overline{A}, \overline{B}$, then U is said to be *symmetric* ; and if $\mathsf{U}(\overline{A}, \overline{B}) = -\mathsf{U}(\overline{B}, \overline{A}), \forall \overline{A}, \overline{B}$, it is *antisymmetric* .

Note also, that if we insert only some of the arguments into this tensor T,

$$\mathsf{T}(\widetilde{\omega}, \widetilde{\cdot}, \overline{\cdot}),$$

then we obtain an object that can take a single one-form and a single vector, and map them into a number; in other words, we have a $\binom{1}{1}$ tensor. If we fill in a further argument

$$\overline{V} = T(\widetilde{\omega}, \widetilde{\cdot}, \overline{A})$$

then we obtain an object with a single one-form argument, which is to say, a vector.

As I said earlier in the section, a vector maps a one-form into a number, and a one-form maps a vector into a number. Thus, for arbitrary $\overline{A}$ and $\widetilde{p}$, both $\overline{A}(\widetilde{p})$ and $\widetilde{p}(\overline{A})$ are numbers. There is nothing in the definition that requires them to be the *same* number, but in GR we will mutually restrict these two functions by requiring that the two numbers are the same in fact. Thus

$$\widetilde{p}(\overline{A}) = \overline{A}(\widetilde{p}) \equiv \langle \widetilde{p}, \overline{A} \rangle, \quad \forall \widetilde{p}, \overline{A} \qquad \text{[in GR]} \tag{2.2}$$

where the notation $\langle \cdot, \cdot \rangle$ emphasises the symmetry of this operation. This combination of the two objects is known as the *contraction* of $\widetilde{p}$ with $\overline{A}$. There are some further remarks about components at the end of Section 2.2.5.

2.2.2 Examples of Tensors

This description of tensors is very abstract, so we need some examples promptly. In this section, we introduce a representation, in terms of row and column vectors, of the structures previously defined. In Section 2.3 we will describe some more representations.

The most immediate example of a vector is the column vector you are familiar with, and the one-forms that correspond to it are simply row-vectors.

$$\begin{aligned} \widetilde{p} &= (p_1, p_2), \\ \overline{A} &= \begin{pmatrix} A^1 \\ A^2 \end{pmatrix}, \\ \langle \widetilde{p}, \overline{A} \rangle &= (p_1, p_2) \begin{pmatrix} A^1 \\ A^2 \end{pmatrix} = p_1 A^1 + p_2 A^2. \end{aligned} \tag{2.3}$$

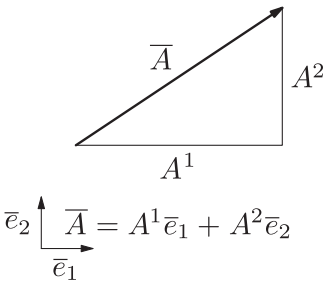

Figure 2.1 A vector.

Here, we see the one-form $\widetilde{p}$ and vector $\overline{A}$ contracting to form a number, by the usual rules of matrix multiplication. Or we can see $\widetilde{p}$ as a real-valued function over vectors, mapping them to numbers, and similarly $\overline{A}$, a real-valued function over one-forms. In this equation, we have *chosen* to define $\langle\widetilde{p}, \overline{A}\rangle$ using the familiar mechanism of matrix multiplication; the definitions of $\overline{A}(\widetilde{p})$ and $\widetilde{p}(\overline{A})$ then come for free, using the equivalences of Eq. (2.2) (I have written the vector components with raised indexes in order to be consistent with the notation introduced in Section 2.2.5; note, by the way, that the vector illustrated in Figure 2.1 is not anchored to the origin – it is not a 'position vector', since that is a thing that would change on any change of origin).

How about tensors of higher rank? Easy: matching the row and column vectors from this section, a square matrix

$$\mathsf{T} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

is a function that takes one one-form and one vector, and maps them to a number, which is to say it is a $\binom{1}{1}$ tensor. If we supply only one of the arguments, to get $\mathsf{T}\overline{A}$, we get an object that has a single one-form argument, which is to say, another vector.

In this specific context, every 2×2 matrix is a $\binom{1}{1}$ tensor, in the sense that it can be contracted with one one-form and one vector. In Section 2.2.5, we will discover that any tensor has a set of components that may be written as a matrix. Do not fall into the trap, however, of thinking that a tensor is 'just' a matrix, or that an arbitrary set of numbers necessarily corresponds, in general, to some tensor. The numbers that are the components of the tensor in some coordinate basis are the results of contracting the tensor with that basis, and as the tensor changes from point to point in the space, or if you change the basis, the components will change systematically. Indeed, the coordinate-based approach to differential geometry, as exemplified by

Rindler (2006), *defines* tensors by requiring that their components change in a systematic way on a change of basis (this approach seemed arbitrary to the point of perversity, when I first learned GR by this route).

As you are aware, there are many quantities in physics that are modelled by an object that has direction and magnitude – for example velocity, force, or angular momentum; if they additionally have the property that they are additive, in the way that two velocities added together make another velocity, then they may be modelled specifically by a vector or one-form (and as we will learn in Section 2.3.1, these are almost indistinguishable in euclidean space, though the distinction is hinted at in mentions of 'pseudovectors' or the occasionally odd behaviour of cross products). There are fewer things that are naturally modelled by higher-rank tensors.

The *inertia tensor* is a rank-2 tensor, In, which, when given an angular velocity vector $\overline{\omega}$, produces the angular momentum $\overline{L}$; or in tensor terms $\overline{L} = \mathsf{In}(\overline{\omega}, \overline{\cdot})$. If we supply $\overline{\omega}$ as the other argument, then we get a quantity T, the kinetic energy, such that $2T = \overline{\omega} \cdot \overline{L} = \mathsf{In}(\overline{\omega}, \overline{\omega})$, and writing $\overline{\omega} = \omega \overline{n}$ and $I = \mathsf{In}(\overline{n}, \overline{n})$, we have[1] the familiar $T = I\omega^2/2$.

In continuum mechanics, the *Cauchy stress tensor* describes the stresses within a body. Given a real or imaginary surface within the body, indicated by a normal $\widetilde{n}$, the stress tensor σ determines the magnitude and direction of the force per unit area experienced by that surface, via $\overline{F} = \sigma(\widetilde{n}, \widetilde{\cdot})$. If we supply this with a (one-form) displacement $\widetilde{s}$, then we find the scalar magnitude of the work done per unit area: $\overline{F}(\widetilde{s}) = \sigma(\widetilde{n}, \widetilde{s})$. Thus the stress tensor takes *two* geometrical objects as arguments, and turns them into a number.[2]

Can we form tensors of other ranks? We can – recall that they are simply a function of some number of vectors and one-forms – as long as we have some way of defining a value for the function, and presumably some physical motivation for wanting to do so.

We can also construct tensors of arbitrary rank by using the *outer product* of multiple vectors or one-forms (this is sometimes also known as the *direct product* or tensor product). We won't actually use this mechanism until we get to Chapter 4, but it's convenient to introduce it here.

[1] See for example Goldstein (2001). Note that here we have elided the distinction between vectors and one-forms, since the distinction does not matter in the euclidean space where we normally care about the inertia tensor.

[2] There are multiple ways of describing forces and displacements in terms of vectors and one-forms (supposing that we are careful enough to care about the distinction between them), and the consequent rank of σ. Every account of continuum mechanics seems to make its own choices here: this variety of 'accents' serves to remind us that mathematics is a way that we have of *describing* nature, and not the same thing as nature itself.

If we have vectors $\overline{V}$ and $\overline{W}$, then we can form a $\binom{2}{0}$ tensor written $\overline{V} \otimes \overline{W}$, the value of which on the one-forms $\widetilde{p}$ and $\widetilde{q}$ is *defined* to be

$$(\overline{V} \otimes \overline{W})(\widetilde{p}, \widetilde{q}) \equiv \overline{V}(\widetilde{p}) \times \overline{W}(\widetilde{q}).$$

This object $\overline{V} \otimes \overline{W}$ is known as the outer product of the two vectors; see Schutz, section 3.4. For example, given two column vectors $\overline{A}$ and $\overline{B}$, the object

$$\overline{A} \otimes \overline{B} = \begin{pmatrix} A^1 \\ A^2 \end{pmatrix} \otimes \begin{pmatrix} B^1 \\ B^2 \end{pmatrix}$$

is a $\binom{2}{0}$ tensor whose value when applied to the two one-forms $\widetilde{p}$ and $\widetilde{q}$ is

$$\begin{aligned} (\overline{A} \otimes \overline{B})(\widetilde{p}, \widetilde{q}) &= \overline{A}(\widetilde{p}) \times \overline{B}(\widetilde{q}) \\ &= (p_1, p_2) \begin{pmatrix} A^1 \\ A^2 \end{pmatrix} \times (q_1, q_2) \begin{pmatrix} B^1 \\ B^2 \end{pmatrix} \\ &= (p_1 A^1 + p_2 A^2) \times (q_1 B^1 + q_2 B^2). \end{aligned}$$

In a similar way, we can use the outer product to form objects of other ranks from suitable combinations of vectors and one-forms. Not all tensors are necessarily outer products, though all tensors can be represented as a sum of outer products.

2.2.3 Fields

We will often want to refer to a scalar-, vector- or tensor-field. A *field* is just a function, in the sense that it maps one space to another, but in this book we restrict the term 'field' to the case of a tensor-valued function, where the domain is a physical space or space-time. That is, a field is a rule that associates a number, or some higher-rank tensor, with each point in space or in space-time. Air pressure is an example of a scalar field (each point in 3-d space has a number associated with it), and the electric and magnetic fields, **E** and **B**, are vector fields (associating a vector with each point in 3-d space).

2.2.4 Visualisation of Vectors and One-Forms

We can visualise vectors straightforwardly as arrows, having both a magnitude and a direction. In order to combine one vector with another, however, we need to add further rules, defining something like the dot product and thus – as we will soon learn – introducing concepts such as the metric (Section 2.2.6).

How do we visualise one-forms in such a way that we distinguish them from vectors, and in such a way that we can visualise (metric-free) operations such as the contraction of a vector and a one-form?

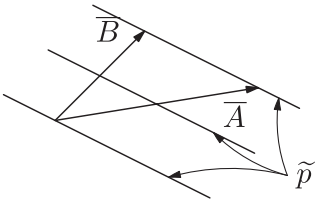

Figure 2.2 Contraction of 2-d vectors and one-form.

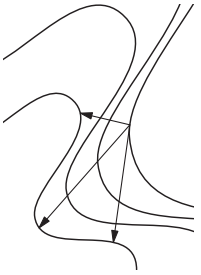

Figure 2.3 Contraction: contours on a map.

The most common way is to visualise a one-form as a set of planes in the appropriate space. Such a structure picks out a direction – the direction perpendicular to the planes – and a magnitude that increases as the separation between the planes *decreases* . The contraction between a vector and a one-form thus visualised is the number of the one-form planes that the vector crosses.

In Figure 2.2, we see two different vectors and one one-form, $\widetilde{p}$. Although the two vectors are of different lengths (though we don't 'know' this yet, since we haven't yet talked about a metric and thus have no notion of 'length'), their contraction with the one-form is the same, namely 2.

You may already be familiar with this picture, if you are familiar with the notion of contours on a map. These show the gradient of the surface they are mapping, with the property that the closer the coutours are together, the larger is the gradient. The three vectors shown in Figure 2.3, which might be different paths up the hillside, have the same contraction – the path climbs three units – even though the three vectors have rather different lengths. When we look at the contours, we are seeing a one-form *field* , with the one-form having different values, both magnitude and direction, at different points in the space. The direction of the gradient always points to higher values.

We will see in Section 3.1.3 that the natural definition of the gradient of a function does indeed turn out to be a one-form.

In Figure 2.4, we this time see a 3-d vector crossing three 2-d planes. Note that, just as you should think of a vector as having a direction and magnitude at

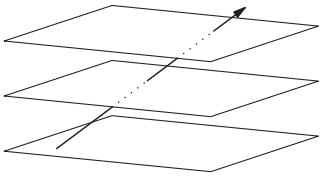

Figure 2.4 A vector contracted with one-form planes.

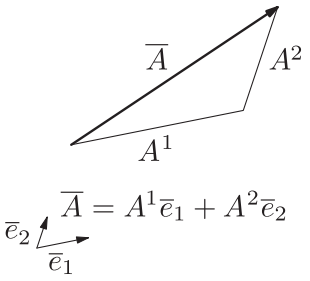

Figure 2.5 An oblique basis.

a point, rather than joining two separated points in space, you should think of a one-form as having a direction and magnitude at a point, and not consisting of actually separate planes.

With this visualisation, it is natural to talk of $\overline{A}$ and $\widetilde{p}$ as *geometrical objects*. When we do so, we are stressing the distinction between, firstly, $\overline{A}$ and $\widetilde{p}$ as abstract objects and, secondly, their numerical components with respect to a basis. This is what we meant when we talked, in Section 1.1, about physical laws depending only on geometrical objects, and not on their components with respect to a set of basis vectors that we introduce only for our mensural convenience.

2.2.5 Components

I said, above, that the set of $\binom{M}{N}$ tensors formed a vector space. Specifically, that includes the sets of vectors and one-forms. From Section 2.1.1, this means that we can find a set of n *basis vectors* $\{\overline{e}_i\}$ and basis one-forms $\{\widetilde{\omega}^i\}$ (this is supposing that the domains of the arguments to our tensors all have the same dimensionality, n; this is not a fundamental property of tensors, but it is true in all the use we make of them, and so this avoids unnecessary complication).

Armed with a set of basis vectors and one-forms, we can write a vector $\overline{A}$ and one-form $\widetilde{p}$ in components as

$$\overline{A} = \sum_i A^i \overline{e}_i; \qquad \widetilde{p} = \sum_i p_i \widetilde{\omega}^i.$$

See Figure 2.1 and Figure 2.5. Crucially, these components are *not* intrinsic to the geometrical objects that $\overline{A}$ and $\widetilde{p}$ represent, but instead depend on the vector or one-form basis that we select. *It is absolutely vital* that you fully appreciate that if you change the basis, you change the components of a vector or one-form (or any tensor) with respect to that basis, *but* the underlying geometrical object, $\overline{A}$ or $\widetilde{p}$ or T, does not change. Though this remark seems obvious now, dealing with it in general is what much of the complication of differential geometry is about.

Note the (purely conventional) positions of the indexes for these basis vectors and one-forms, and for the components: the components of vectors have raised indexes, and the components of one-forms have lowered indexes. This convention allows us to define an extremely useful notational shortcut, which allows us in turn to avoid writing hundreds of summation signs:

Einstein summation convention: whenever we see an index repeated in an expression, once raised and once lowered, we are to understand a summation over that index.

Thus:

$$A^i\overline{e}_i \equiv \sum_i A^i\overline{e}_i; \qquad p_i\widetilde{\omega}^i \equiv \sum_i p_i\widetilde{\omega}^i.$$

We have illustrated this for components and vectors here, but it will apply quite generally in the following rules for working with components:

Here are the rules for working with components:

1. In any expression, there must be at most two of each index, one raised and one lowered. If you have more than two, or have both raised or lowered, you've made a mistake. Any indexes 'left over' after contraction tell you the rank of the object of which this is the component.
2. The components are just numbers, and so, as you learned in primary school, it doesn't matter what order you multiply them (they don't commute with differential signs, though). If they are the components of a field, then the components, as well as the basis vectors, may vary across the space.
3. The indexes are arbitrary – you can always replace an index letter with another one, as long as you do it consistently. That is, $p_iA^i = A^jp_j$, and $p_iq_jT^{ij} = p_jq_iT^{ji} = p_kq_iT^{ki}$ (though $p_kq_iT^{ki} \neq p_kq_iT^{ik}$ in general, unless the tensor T is symmetric).

What happens if we apply $\widetilde{p}$, say, to one of the basis vectors? We have

$$\widetilde{p}(\overline{e}_j) = p_i\widetilde{\omega}^i(\overline{e}_j). \tag{2.4}$$

In principle, we know nothing about the *number* $\tilde{\omega}^i(\bar{e}_j)$, since we are at liberty to make completely independent choices of the vector and one-form bases. However, we can save ourselves ridiculous amounts of trouble by making a wise choice, and we will always choose one-form bases to have the property

$$\langle \tilde{\omega}^i, \bar{e}_j \rangle = \tilde{\omega}^i(\bar{e}_j) = \bar{e}_j(\tilde{\omega}^i) = \delta^i_j \equiv \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \tag{2.5}$$

A one-form basis with this property is said to be *dual* to the vector basis. Returning to Eq. (2.4), therefore, we find

$$\tilde{p}(\bar{e}_j) = p_i \tilde{\omega}^i(\bar{e}_j) = p_i \delta^i{}_j = p_j. \tag{2.6}$$

Thus in the one-form basis that is dual to the vector basis $\{\bar{e}_j\}$, the arbitrary one-form $\tilde{p}$ has components $p_j = \tilde{p}(\bar{e}_j)$.

Similarly, we can apply the vector $\bar{A}$ to the one-form basis $\{\tilde{\omega}^i\}$, and obtain

$$\bar{A}(\tilde{\omega}^i) = A^j \bar{e}_j(\tilde{\omega}^i) = A^j \delta_j{}^i = A^i.$$

In exactly the same way, we can apply the tensor T to the basis vectors and one-forms, and obtain

$$\mathsf{T}(\tilde{\omega}^i, \tilde{\omega}^j, \bar{e}_k) = T^{ij}{}_k. \tag{2.7}$$

The set of $n \times n \times n$ *numbers* $\{T^{ij}{}_k\}$ are the *components* of the tensor T in the basis $\{\bar{e}_i\}$ and its dual $\{\tilde{\omega}^j\}$. We will generally denote the vector $\bar{A}$ by simply writing 'A^i', denote $\tilde{p}$ by 'p_i', and the $\binom{2}{1}$ tensor T by '$T^{ij}{}_k$'. Because of the index convention, we will always know what sort of object we are referring to by whether the indexes are raised or lowered: the components of vectors always have their indexes raised, and the components of one-forms always have their indexes lowered.

Notice the pattern: the components of vectors have raised indexes, but the basis vectors themselves are in contrast written with lowered indexes, and *vice versa* for one-forms. It is this notational convention that allows us to take advantage of the Einstein summation convention when writing a vector as $\bar{A} = A^i \bar{e}_i$ or $\tilde{p} = p_i \tilde{\omega}^i$.

We can, obviously, find the components of the basis vectors and one-forms by exactly this method, and find

$$\begin{aligned} \bar{e}_1 &\to (1, 0, \ldots, 0) \\ \bar{e}_2 &\to (0, 1, \ldots, 0) \\ &\vdots \\ \bar{e}_n &\to (0, 0, \ldots, 1) \end{aligned} \tag{2.8}$$

where the numbers on the right-hand-side are the components in the vector basis, and

$$\begin{aligned} \widetilde{\omega}^1 &\to (1,0,\ldots,0) \\ \widetilde{\omega}^2 &\to (0,1,\ldots,0) \\ &\vdots \\ \widetilde{\omega}^n &\to (0,0,\ldots,1) \end{aligned} \tag{2.9}$$

where the components are in the one-form basis. Make sure you understand why Eqs. (2.8) and (2.9) are 'obvious'.

So what is the value of the expression $\widetilde{p}(\overline{A})$ in components? By linearity,

$$\widetilde{p}(\overline{A}) = p_i \widetilde{\omega}^i (A^j \overline{e}_j) = p_i A^j \widetilde{\omega}^i (\overline{e}_j) = p_i A^j \delta^i{}_j = p_i A^i .$$

This is the *contraction* of $\widetilde{p}$ with $\overline{A}$. Note particularly that, since $\widetilde{p}$ and $\overline{A}$ are basis-independent, geometrical objects – or quite separately, since $\widetilde{p}(\overline{A})$ is a pure number – the *number* $p_i A^i$ is basis-independent also, even though the numbers p_i and A^i are separately basis-dependent.

Similarly, contracting the $\binom{2}{1}$ tensor T with one-forms $\widetilde{p}$, $\widetilde{q}$ and vector $\overline{A}$, we obtain the number

$$\mathsf{T}(\widetilde{p}, \widetilde{q}, \overline{A}) = p_i q_j A^k T^{ij}{}_k .$$

If we contract it instead with just the one-forms, we obtain the object $\mathsf{T}(\widetilde{p}, \widetilde{q}, \overline{\ \cdot\ })$, which is a one-form (since it maps a single vector to a number) with components

$$\mathsf{T}(\widetilde{p}, \widetilde{q}, \overline{\ \cdot\ })_k = p_i q_j T^{ij}{}_k$$

and the solitary unmatched lower index k on the right-hand-side indicates (or rather confirms) that this object is a one-form. The indexes are staggered so that we keep track of which argument they correspond to. I noted before that the two tensors $\mathsf{T}(\overline{\ \cdot\ }, \widetilde{\ \cdot\ })$ and $\mathsf{T}(\widetilde{\ \cdot\ }, \overline{\ \cdot\ })$ are different tensors: *if* the tensor is symmetric, then $\mathsf{T}(\overline{e}_i, \widetilde{\omega}^j) = \mathsf{T}(\widetilde{\omega}^j, \overline{e}_i)$, and thus $T_i{}^j = T^j{}_i$, but we cannot simply assume this.

We can also form the contraction of a tensor, by pairing up a vector- and one-form-shaped argument, to make a tensor of rank two smaller. Considering the $\binom{2}{1}$ tensor T as before, we can define a new tensor S as

$$\mathsf{S}(\widetilde{\ \cdot\ }) = \mathsf{T}(\widetilde{\ \cdot\ }, \widetilde{\omega}^j, \overline{e}_j), \tag{2.10}$$

where the indexes j are summed over as usual. Pairing up different slots in T would produce different contracted tensors S. In component form, this is

$$S^i = T^{ij}{}_j . \tag{2.11}$$

In fact, the operation of contraction is *defined* , as here, on tensors – it takes a tensor into a tensor of rank two less. Earlier in this section, we defined contraction as an operation on a vector and one-form; we can now see that this is just a consequence of this definition: defining $\mathsf{S} = \widetilde{p} \otimes \overline{A}$, we can immediately form the contraction of this tensor into a scalar, $S_i{}^i = p_i A^i$, and this is what we defined as the 'contraction of $\widetilde{p}$ and $\overline{A}$'. MTW have a useful discussion of contraction in their section 3.5, as part of a longer discussion of ways of producing new tensors from old. [Exercises 2.3–2.5]

2.2.6 The Metric Tensor

One thing we do not have yet is any notion of distance, but we can supply that very easily, by picking a symmetric $\binom{0}{2}$ tensor g, and calling that the *metric tensor* , or just 'the metric'.

The metric allows us to define an inner product between vectors (which in other contexts we might call a scalar product or dot product). The *inner product* between two vectors $\overline{A}$ and $\overline{B}$ is the scalar

$$\overline{A} \cdot \overline{B} = \mathsf{g}(\overline{A}, \overline{B})$$

We can define the length of a vector as the square root of its inner product with itself: $|\overline{A}|^2 = \mathsf{g}(\overline{A}, \overline{A})$. We can also use this to define an angle θ between two vectors via

$$\overline{A} \cdot \overline{B} = |\overline{A}||\overline{B}| \cos\theta.$$

Note that since g is a tensor, it is frame independent, so that the length $|\overline{A}|$ and angle θ must be frame-independent quantities also.

We can find the components of the metric tensor in the same way we can find the components of any earlier tensor:

$$\mathsf{g}(\overline{e}_i, \overline{e}_j) = g_{ij}. \tag{2.12}$$

As well as giving us a notion of length, the metric tensor allows us to define a mapping between vectors and one-forms. Since it is a $\binom{0}{2}$ tensor, it is a thing which takes two vectors and turns them into a number. If instead we only supply a single vector $\overline{A} = A^i \overline{e}_i$ to the metric, we have a thing which takes one further vector and turns it into a number; but this is just a one-form, which we will write as $\widetilde{A}$:

$$\widetilde{A} = \mathsf{g}(\overline{A}, \cdot\,) = \mathsf{g}(\cdot\,, \overline{A}). \tag{2.13}$$

That is, for any vector $\overline{A}$, we have found a way of picking out a single associated one-form, written $\widetilde{A}$. What are the components of this one-form? Easy:

$$\begin{aligned} A_i = \widetilde{A}(\overline{e}_i) &= \mathsf{g}(\overline{e}_i, \overline{A}) \\ &= \mathsf{g}(\overline{e}_i, A^j \overline{e}_j) \\ &= A^j \mathsf{g}(\overline{e}_i, \overline{e}_j) \\ &= g_{ij} A^j, \end{aligned} \tag{2.14}$$

from Eq. (2.12) above. That is, the metric tensor can also be regarded as an '*index lowering* ' operator.

Can we do this trick in the other direction, defining a $\binom{2}{0}$ tensor that takes two one-forms as arguments and turns them into a number? Yes we can, and the natural way to do it is via the tensor's components.

The set of numbers g_{ij} is, at one level, just a matrix. Thus if it is non-singular (and we will always assume that the metric is non-singular), this matrix has an inverse, and we can take the components of the tensor we're looking for, g^{ij}, to be the components of this inverse. That means nothing other than

$$g^{ij} g_{jk} = g^i{}_k = \delta^i{}_k, \tag{2.15}$$

We will refer to the tensors corresponding to g^{ij}, $g^i{}_j$ and g_{ij} indiscriminately as 'the metric'.

What happens if we apply g^{ij} to the one-form components A_j?

$$g^{ij} A_j = g^{ij} g_{jk} A^k = \delta^i{}_k A^k = A^i, \tag{2.16}$$

so that the metric can raise components as well as lower them.

There is nothing in the discussion above that says that the tensor g has the same value at each point in space-time. In general, g is a tensor field, and the different values of the metric at different points in spacetime are associated with the curvature of that space-time. This is where the physics comes in.

[Exercises 2.6 and 2.7]

2.2.7 Changing Basis

The last very general set of properties we must discover about tensors is what happens when you change your mind about the sets of basis vectors and one-forms (you can't change your mind about just one of them, if they are to remain dual to each other).

We (now) know that if we have a set of basis vectors $\{\overline{e}_i\}$, then we can find the components of an arbitrary vector $\overline{A}$ to be $A^i = \overline{A}(\widetilde{\omega}^i)$, where the $\{\widetilde{\omega}^i\}$ are the set of basis one-forms that are dual to the vectors $\{\overline{e}_i\}$.

But there is nothing special about this basis, and we could equally well have chosen a completely different set $\{\overline{e}_{\bar{\jmath}}\}$ with dual $\{\widetilde{\omega}^{\bar{\jmath}}\}$. With respect to this basis, the *same* vector $\overline{A}$ can be written

$$\overline{A} = A^{\bar{\imath}}\overline{e}_{\bar{\imath}},$$

where, of course, the components are the set of numbers

$$A^{\bar{\imath}} = \overline{A}(\widetilde{\omega}^{\bar{\imath}}).$$

It's important to remember that A^i and $A^{\bar{\imath}}$ are different (sets of) numbers because they refer to different bases $\{\overline{e}_i\}$ and $\{\overline{e}_{\bar{\imath}}\}$, but that they correspond to the *same* underlying vector $\overline{A}$ (this is why we distinguish the symbols by putting a bar on the index i rather than on the base symbol $\overline{e}$ or $\overline{A}$ – this does look odd, I know, but it ends up being notationally tidier than the various alternatives).[3]

Since both these sets of components represent the *same* underlying object $\overline{A}$, we naturally expect that they are related to each other, and it is easy to write down that relation. From before

$$\begin{aligned} A^{\bar{\imath}} &= \overline{A}(\widetilde{\omega}^{\bar{\imath}}) \\ &= A^i\overline{e}_i(\widetilde{\omega}^{\bar{\imath}}) \\ &= \Lambda^{\bar{\imath}}_i A^i, \end{aligned} \tag{2.17}$$

where we have written the *transformation matrix* Λ as

$$\Lambda^{\bar{\imath}}_i \equiv \overline{e}_i(\widetilde{\omega}^{\bar{\imath}}) \equiv \widetilde{\omega}^{\bar{\imath}}(\overline{e}_i). \tag{2.18}$$

Note that Λ is a *matrix*, not a tensor – there's no underlying geometrical object, and we have consequently not staggered its indexes (see also the remarks on this at the end of Section 2.3.2). Also, note that indexes i and $\imath$ are completely distinct from each other, and arbitrary, and we are using the similarity of symbols just to emphasise the symmetry of the operation. Exactly analogously, the components of a one-form $\widetilde{p}$ transform as

$$p_{\bar{\imath}} = \Lambda^i_{\bar{\imath}} p_i, \tag{2.19}$$

where the transformation matrix is

$$\Lambda^i_{\bar{\imath}} \equiv \widetilde{\omega}^i(\overline{e}_{\bar{\imath}}). \tag{2.20}$$

[3] Notation: it is slightly more common to distinguish the bases by a prime on the index, as in $\overline{e}_{i'}$, or even sometimes a hat, $\overline{e}_{\hat{\imath}}$. I prefer the overbar on the practical grounds that it seems easier to distinguish in handwriting – try writing '$;i'$' three times quickly.

Since the vector $\overline{A}$ is the same in both coordinate systems, we must have

$$\begin{aligned}\overline{A} &= A^i \overline{e}_i = A^{\bar{\imath}} \overline{e}_{\bar{\imath}} \\ &= A^j \Lambda^{\bar{\imath}}_j E^k_{\bar{\imath}} \overline{e}_k,\end{aligned} \tag{2.21}$$

where we write $E^k_{\bar{\imath}}$ as the (initially unknown) components of vector $\overline{e}_{\bar{\imath}}$ in the basis $\{\overline{e}_k\}$ (ie $\overline{e}_{\bar{\imath}} = E^k_{\bar{\imath}} \overline{e}_k$). This immediately requires that $\Lambda^{\bar{\imath}}_j E^k_{\bar{\imath}} = \delta^k_j$, and thus that the matrix E must be the inverse of the matrix Λ. Now we're going to find the same expression by a different route.

The components of $\widetilde{\omega}^i$ in the barred basis are (as usual) $\widetilde{\omega}^i(\overline{e}_{\bar{\imath}})$, which is equal to $\Lambda^i_{\bar{\imath}}$ (from Eq. (2.20)), and similarly the components of $\overline{e}_j$ are $\overline{e}_j(\widetilde{\omega}^{\bar{\jmath}}) = \Lambda^{\bar{\jmath}}_j$ (from Eq. (2.18)). So now look at the contraction $\widetilde{\omega}^i(\overline{e}_j)$ in two coordinate systems:

$$\delta^i_j = \widetilde{\omega}^i(\overline{e}_j) = \Lambda^i_{\bar{\imath}} \widetilde{\omega}^{\bar{\imath}}(\Lambda^{\bar{\jmath}}_j \overline{e}_{\bar{\jmath}}) = \Lambda^i_{\bar{\imath}} \Lambda^{\bar{\jmath}}_j \widetilde{\omega}^{\bar{\imath}}(\overline{e}_{\bar{\jmath}}) = \Lambda^i_{\bar{\imath}} \Lambda^{\bar{\jmath}}_j \delta^{\bar{\imath}}_{\bar{\jmath}} = \Lambda^i_{\bar{\imath}} \Lambda^{\bar{\imath}}_j. \tag{2.22}$$

Thus $\Lambda^i_{\bar{\imath}}$ and $\Lambda^{\bar{\imath}}_j$ are matrix inverses of each other; so we must have $E^i_{\bar{\imath}} = \Lambda^i_{\bar{\imath}}$, and

$$\overline{e}_{\bar{\imath}} = \Lambda^i_{\bar{\imath}} \overline{e}_i. \tag{2.23}$$

The results of Exercise 2.10 amplify this point.

Here, it has been convenient to introduce basis transformations by focusing on the transformation of the *components* of vectors and one-forms, in Eq. (2.17). We could alternatively introduce them by focusing on the transformation of the basis vectors and one-forms themselves, and this is the approach used in the discussion of basis transformations in Section 3.1.4.

Finally, it is easy to show that the components of other tensors transform in what might be the expected pattern, with each unbarred index in one coordinate system matched by a suitable Λ term. For example the components of our $\binom{2}{1}$ tensor T will transform as

$$T^{\bar{\imath}\bar{\jmath}}{}_{\bar{k}} = \Lambda^{\bar{\imath}}_i \Lambda^{\bar{\jmath}}_j \Lambda^k_{\bar{k}} T^{ij}{}_k. \tag{2.24}$$

The $\delta^i{}_j$ are the components of the Kronecker *tensor* . As a $\binom{1}{1}$ tensor, this maps vectors to vectors, or one-forms to one-forms, via the identity map: $\delta(\overline{A}, \widetilde{\cdot})^i = \delta^i{}_j A^j = A^i$. It also has the property that its components are the same in every basis: $\Lambda^j_{\bar{\jmath}} \Lambda^{\bar{\imath}}_i \delta^i_j = \Lambda^j_{\bar{\jmath}} \Lambda^{\bar{\imath}}_j = \delta^{\bar{\imath}}_{\bar{\jmath}}$. The property Eq. (2.5) means that tensors $\delta(\overline{\cdot}, \widetilde{\cdot})$ and $\delta(\widetilde{\cdot}, \overline{\cdot})$ are equal, which is why we could afford to be casual, at the end of Section 2.1.1, about whether we write $\delta^i{}_j$, $\delta_i{}^j$ or indeed δ^i_j. Further, Eq. (2.15) shows that this tensor is the same as the metric tensor with one index raised and one lowered or, equivalently, that the components of the $\binom{0}{2}$ Kronecker tensor are $\delta_{ij} = g_{ik} \delta^k{}_j = g_{ij}$. [Exercises 2.8–2.11]

2.2.8 Some Misconceptions about Coordinates and Vectors

This section has a dangerous-bend marker, not because it introduces extra material, but because it should be approached with caution. It addresses some misconceptions about coordinates that have cropped up at various times. The danger of mentioning these, of course, is that if you learn about a misconception that hadn't occurred to you before, it could make things worse! Also, I make forward references to material introduced in Chapter 3, so this section is probably more useful at revision time than on your first read through. That said, if you're getting confused about the transformation material that we've discussed so far (and you're not alone), then the following remarks might help.

There might be a temptation to write down something that appears to be the 'coordinate form':

$$x^{\bar{i}} = \Lambda^{\bar{i}}_{i} x^{i} \qquad \text{(meaningless)}. \tag{2.25}$$

This looks a bit like Eq. (2.23), and a bit like Eq. (2.17); it feels like it should be describing the transformation of the $\{x^i\}$ coordinates into the $\{x^{\bar{i}}\}$ ones, so that it may appear to be the analogue of Eq. (2.27). It's none of these things, however.

Notice that we haven't, so far, talked of *coordinates* at all. When we talked of components in Section 2.2.5, and of changing bases in Section 2.2.7 (which of course we understand to be a change of coordinates), we did so by talking about *basis vectors*, $\bar{e}_i$. This is like talking about cartesian coordinates by talking exclusively about the basis vectors **i**, **j**, and **k**, and avoiding talking about $\{x, y, z\}$.

In Section 3.1.1, we introduce coordinates as *functions* on the manifold, $\{x^i : M \to \mathbb{R}\}$, and in Eq. (3.4) we define the basis vectors associated with them as $\bar{e}_i = \partial/\partial x^i$ (see also the discussion at the end of Section 3.1.4). Thus Eq. (2.25) is suggesting that these coordinate functions are linear combinations of each other; that will generally not be true, and it is possible to get very confused in Exercise 3.3, for example, by thinking in this way. It is tempting to look at Eq. (2.25) and interpret the x^i as components of the basis vectors, or something like that, but the real relationship is the other way around: the basis vectors are derived from the coordinate functions, and show the way in which the coordinate functions change as you move around the manifold. The components of the basis vectors are very simple – see Eq. (2.8).

It's also worth stressing, once we're talking about misconceptions, that *neither position vectors, nor connecting vectors, are 'vectors' in the sense introduced in this part of the notes*. In a flat space, such as the euclidean space

of our intuitions, or the flat space of Special Relativity, the difference between them disappears or, to put it another way, there is a one–to-one correspondence between 'a vector in the space' and 'the difference between two positions' (which is what a difference vector is). In a curved space, it's useful to talk about the former (and we do, at length), but the latter won't often have much physical meaning. It is because of this correspondence that we can easily 'parallel transport' vectors everywhere in a flat space (see Section 3.3.2), which means we have been able, at earlier stages in our education, to define vector differentiation without having to think about it very hard.

If you think of a *vector field* – that is, a field of vectors, such as you tend to imagine in the case of the electric field – then the things you imagine existing at each point in space-time are straightforwardly vectors. That is, they're a thing with magnitude and direction (but not spatial extent), defined at each point.

2.3 Examples of Bases and Transformations

So far, so abstract. By now, we are long overdue for some illustrations.

2.3.1 Flat Cartesian Space

Consider the natural vectors on the euclidean plane – that is, the vectors you learned about in school. The obvious thing to do is to pick our basis to be the unit vectors along the x and y axes: $\overline{e}_1 = \overline{e}_x$ and $\overline{e}_2 = \overline{e}_y$. That means that the vector $\overline{A}$, for example, which points from one point to a point two units along and one up, can be written as $\overline{A} = 2\overline{e}_1 + 1\overline{e}_2$, or in other words that it has components $A^1 = 2$, $A^2 = 1$. We have chosen these basis vectors to be the usual orthonormal ones: however, we are *not* required to do this by anything in Section 2.2, and indeed we cannot even say this at this stage, because we have not (yet) defined a metric, and so we have no inner product, so that the ideas of 'orthogonal' and 'unit' do not yet exist.

What are the one-forms in this space? Possibly surprisingly, there is nothing in Section 2.2 that tells us what they are, so that we can pick anything we like as a one-form in the euclidean plane, as long as that one-form-thing obeys the axioms of a vector space (Section 2.1.1), and as long as whatever rule we devise for contracting vectors and one-form-things conforms to the constraint of Eq. (2.2).

For one-forms, then, and as suggested in Section 2.2.4, we'll choose sets of planes all parallel to each other, with the property that if we 'double' a one-form, then the spacing between the planes halves (recall Figure 2.4). For our

contraction rule, $\langle \cdot , \cdot \rangle$, we'll choose: 'the number $\langle \tilde{p}, \overline{A} \rangle$ is the number of planes of the one-form $\tilde{p}$ that the vector $\overline{A}$ passes through'. If the duality property Eq. (2.5) is to hold, then this fixes the 'planes' of $\tilde{\omega}^1$ to be lines perpendicular to the x-axis, one unit apart, and the planes of $\tilde{\omega}^2$ to be similarly perpendicular to the y-axis.

For our metric, we can choose simply

$$g_{ij} = g^{ij} = g^{i}{}_{j} = \delta^{i}_{j} \qquad \text{[cartesian coordinates]}. \tag{2.26}$$

This means that the length-squared of the vector $\overline{A} = 2\overline{e}_1 + 1\overline{e}_2$ is

$$\mathsf{g}(\overline{A}, \overline{A}) = g_{ij} A^{i} A^{j} = A^1 A^1 + A^2 A^2 = (2)^2 + (1)^2 = 5,$$

which corresponds to our familiar value for this, from Pythagoras' theorem.

The other interesting thing about this metric is that, when we use it to lower the indexes of an arbitrary vector $\overline{A}$, we find that $A_i = g_{ij} A^j = A^i$. In other words, for this metric (the natural one for this flat space, with orthonormal basis vectors) one-forms and vectors have the *same* components, so that we can no longer really tell the difference between them, and have to work hard to think of them as being separate. *This* is why you have never had to deal with one-forms before, because the distinction between the two things in the space of our normal (mathematical) experience is invisible.

2.3.2 Polar Coordinates

An alternative way of devising vectors for the euclidean plane is to use polar coordinates. It is convenient to introduce these using the transformation equation Eq. (2.17). The radial and tangential basis vectors are

$$\overline{e}_{\bar{1}} = \overline{e}_r = \cos\theta\, \overline{e}_1 + \sin\theta\, \overline{e}_2 \tag{2.27a}$$

$$\overline{e}_{\bar{2}} = \overline{e}_\theta = -r \sin\theta\, \overline{e}_1 + r\cos\theta\, \overline{e}_2, \tag{2.27b}$$

where $\overline{e}_1 = \overline{e}_x$ and $\overline{e}_2 = \overline{e}_y$, as before. Note that these basis vectors vary over the plane, and that although they are orthogonal (though we 'don't know that yet' since we haven't defined a metric), they are not orthonormal. Thus we can write

$$\overline{e}_{\bar{1}} = \overline{e}_r = \Lambda^{1}{}_{\bar{1}} \overline{e}_1 + \Lambda^{2}{}_{\bar{1}} \overline{e}_2$$

$$\overline{e}_{\bar{2}} = \overline{e}_\theta = \Lambda^{1}{}_{\bar{2}} \overline{e}_1 + \Lambda^{2}{}_{\bar{2}} \overline{e}_2,$$

and so discover that

$$\Lambda^{i}{}_{\bar{i}} = \begin{pmatrix} \Lambda^{1}{}_{\bar{1}} & \Lambda^{1}{}_{\bar{2}} \\ \Lambda^{2}{}_{\bar{1}} & \Lambda^{2}{}_{\bar{2}} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}, \tag{2.28}$$

where we have written the matrix Λ so that

$$(\bar{e}_{\bar{1}}, \bar{e}_{\bar{2}}) = (\bar{e}_1, \bar{e}_2) \begin{pmatrix} \Lambda^1{}_{\bar{1}} & \Lambda^1{}_{\bar{2}} \\ \Lambda^2{}_{\bar{1}} & \Lambda^2{}_{\bar{2}} \end{pmatrix} \tag{2.29}$$

recovers Eq. (2.23), and *in this section alone* staggered the indexes of Λ to help keep track of the elements of Λ and its inverse, written as matrix expressions. Therefore, if we require $\Lambda^{\bar{\imath}}{}_i \Lambda^i{}_{\bar{\jmath}} = \delta^{\bar{\imath}}_{\bar{\jmath}}$, then we must have

$$\Lambda^{\bar{\imath}}{}_i = \begin{pmatrix} \Lambda^{\bar{1}}{}_1 & \Lambda^{\bar{1}}{}_2 \\ \Lambda^{\bar{2}}{}_1 & \Lambda^{\bar{2}}{}_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta/r & \cos\theta/r \end{pmatrix} \tag{2.30}$$

(you can confirm that if you matrix-multiply Eq. (2.30) by Eq. (2.28), then you retrieve the unit matrix, as Eq. (2.22) says you should). Symmetrically with Eq. (2.29), we can now write

$$\begin{pmatrix} A^{\bar{1}} \\ A^{\bar{2}} \end{pmatrix} = \begin{pmatrix} \Lambda^{\bar{1}}{}_1 & \Lambda^{\bar{1}}{}_2 \\ \Lambda^{\bar{2}}{}_1 & \Lambda^{\bar{2}}{}_2 \end{pmatrix} \begin{pmatrix} A^1 \\ A^2 \end{pmatrix}, \tag{2.31}$$

and discover that

$$(\bar{e}_{\bar{1}}, \bar{e}_{\bar{2}}) \begin{pmatrix} A^{\bar{1}} \\ A^{\bar{2}} \end{pmatrix} = (\bar{e}_1, \bar{e}_2) \begin{pmatrix} A^1 \\ A^2 \end{pmatrix},$$

where the contraction of vector and one-form is coordinate-independent.

We therefore know how to transform the components of vectors and one-forms between cartesian and plane polar coordinates. What does the metric tensor ($g_{ij} = \delta_{ij}$ in cartesian coordinates, remember) look like in these new coordinates? The components are just

$$g_{\bar{\imath}\bar{\jmath}} = \Lambda^i{}_{\bar{\imath}} \Lambda^j{}_{\bar{\jmath}} g_{ij},$$

and writing these components out in full, we find

$$g_{\bar{\imath}\bar{\jmath}} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}. \tag{2.32}$$

We see that, even though coordinates (x, y) and (r, θ) are describing the *same* flat space, the metric looks a little more complicated in the polar-coordinate coordinate system than it does in the plain cartesian one, and looking at this out of context, we would have difficulty identifying the space described by Eq. (2.32) as flat euclidean space.

We will see a lot more of this in the parts to come.

The definition of polar coordinates in Eq. (2.27) is not the one you are probably familar with, due to the presence of the r in the transformations. The expressions can be obtained very directly, by choosing as the transformation

$$\bar{e}_r \equiv \Lambda^x_r \bar{e}_x + \Lambda^y_r \bar{e}_y = \frac{\partial x}{\partial r}\bar{e}_x + \frac{\partial y}{\partial r}\bar{e}_y$$
$$\bar{e}_\theta \equiv \Lambda^x_\theta \bar{e}_x + \Lambda^y_\theta \bar{e}_y = \frac{\partial x}{\partial \theta}\bar{e}_x + \frac{\partial y}{\partial \theta}\bar{e}_y.$$

This is known as a *coordinate basis* , since it is generated directly from the relationship between the coordinate functions, $r^2 = x^2 + y^2$ and $\tan\theta = y/x$. Although this is a very natural definition of the new basis vectors, and is the type of transformation we will normally prefer, these basis vectors $\bar{e}_r$ and $\bar{e}_\theta$ are not of unit length, and indeed are of different lengths at different points in the coordinate plane. That is why the usual definition of polar basis vectors is chosen to be $\bar{e}_\theta = (1/r)\partial x/\partial\theta\, \bar{e}_x + (1/r)\partial y/\partial\theta\, \bar{e}_y$, which is a *non-coordinate basis* . See Schutz §5.5 for further discussion.

From Eq. (2.29) we can recover the transformation expression $\bar{e}_{\bar{\imath}} = \Lambda^i_{\bar{\imath}} \bar{e}_i$ by the usual rules of matrix multiplication. Presenting it this way helps emphasise that the object Λ is indeed 'just' a matrix and not anything more exotic; this matrix representation also appears in one or two of the problems. While this may be usefully concrete, I do not believe it to be a very useful way of manipulating these objects in general, because it is easy to get the matrix the wrong way around (this is why the index-summation notation is a useful one). Some authors consistently stagger the indexes of the Λ matrices here, in order to help keep things straight; I think that adds notational intricacy for little practical benefit. I'll write out Λ matrices when the explicitness seems helpful, but if this representation seems obscure, then don't worry about it. You won't be missing anything deep. [Exercise 2.12]

2.3.3 Matrices, Row, and Column Vectors

We can regard the set of all n-component *column* vectors as a vector space (the terminology in this subsection is going to be confusing!). This means that the n-component *row* vectors are the one-forms in the vector space dual to the column vectors. Why? Because a row vector can be contracted with a column vector in the usual way to produce a number, which is exactly the defining relation between vectors and one-forms. Similarly the $n \times n$ matrices (which, since you can add them and multiply them by scalars, are *also* a vector space) are examples of $\binom{1}{1}$ tensors. Why? Because if you contract

a matrix with a column vector (by the usual means of right-multiplying the matrix by the column vector) you get a column vector, and if you contract the row-vector/one-form with a matrix (by the usual means of left-multiplying the matrix by the row-vector), you get another row-vector/one-form. If you both left-multiply a matrix by a row vector, and right-multiply it by a column vector, then you end up, of course, with just a number, in $\mathbb{R}$.

What we have done here is to regard column vectors, row vectors, and square matrices as *representations* of the abstract structures of respectively $\binom{1}{0}$, $\binom{0}{1}$, and $\binom{1}{1}$ tensors (in this approach we can't conveniently find representations of higher-rank tensors). We have done two non-trivial things here: (i) we have selected three vector spaces to play with, and (ii) we have defined 'function application', as required by the definition of a tensor in Section 2.2.1, to be the familiar matrix multiplication. Thus in this representation, any square matrix is a $\binom{1}{1}$ tensor.

2.3.4 Minkowski Space: Vectors in Special Relativity

The final very important example of the ideas of this part is the space of SR: *Minkowski space*. See Schutz, chapter 2, though he comes at the transformation matrix Λ from a slightly different angle.

Here there are four dimensions rather than two, and a basis for the space is formed from $\overline{e}_0 = \overline{e}_t$ and $\overline{e}_{1,2,3} = \overline{e}_{x,y,z}$. As is conventional in SR, we will now use Greek indices for the vectors and one-forms, with the understanding that Greek indices run over $\{0, 1, 2, 3\}$. The metric on this space, in these coordinates, is

$$g_{\mu\nu} = \eta_{\mu\nu} \equiv \text{diag}(-1, 1, 1, 1) \tag{2.33}$$

(note that this convention for the metric, with a signature of $+2$, is the same as in Schutz; some texts choose the opposite convention of $\eta_{\mu\nu} = \text{diag}(+1, -1, -1, -1)$.). See also Section 1.4.3.

Vectors in this space are $\overline{A} = A^\mu \overline{e}_\mu$, and we can use the metric to lower the indexes and form the components in the dual space (which we define in a similar way to the way we defined the dual space in Section 2.3.1). Thus the contraction between a one-form $\widetilde{A}$ and vector $\overline{B}$ is just

$$\begin{aligned}\left\langle \widetilde{A}, \overline{B} \right\rangle &= A_\mu B^\mu = A_0 B^0 + A_1 B^1 + A_2 B^2 + A_3 B^3 \\ &= \eta_{\mu\nu} A^\mu B^\nu = -A^0 B^0 + A^1 B^1 + A^2 B^2 + A^3 B^3.\end{aligned}$$

This last expression should be very familiar to you, since it is exactly the definition of the *norm* of two vectors which was so fundamental, and which seemed so peculiar, in SR.

We can define transformations from these Minkowski-space coordinates to new ones in the same space, by specifying the elements of a transformation matrix Λ (cf. Schutz §2.2). We can do this however we like, but there is a subset of these transformations which are particularly useful, since they result in the metric in the new coordinates having the same form as the metric in the old one, namely $g_{\bar{\mu}\bar{\nu}} = \eta_{\bar{\mu}\bar{\nu}}$. One of the simplest sets of such transformation matrices (parameterised by $0 \leq v \leq 1$) is

$$\Lambda^{\bar{\mu}}_{\mu}(v) = \begin{pmatrix} \gamma & -v\gamma & 0 & 0 \\ -v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2.34}$$

where $\gamma = 1/\sqrt{1 - v^2}$. Again, this should be rather familiar.

[Exercise 2.13]

2.4 Coordinates and Spaces

There are a few points of terminology that are useful to gather here, even though their full significance will not be apparent until later sections.

Note that in the previous section we have distinguished *cartesian coordinates* from a *euclidean space*. The first is the system of rectilinear coordinates you learned about in school, and the second is the flat space of our normal experience (where the ratio of the circumference of a circle to its diameter is π, and so on). In Section 2.3.1, we talked about euclidean space described by cartesian coordinates. In Section 2.3.2, we discussed euclidean space, but not in cartesian coordinates, and in Section 2.3.4 we used rectilinear coordinates to describe a non-euclidean space, Minkowski space, which does not have the same metric as the two previous examples.

Euclidean space and Minkowski space are both *flat spaces*: in each of them, Euclid's parallel postulate holds, and there is a global definition of parallelism in the sense that if a vector is moved parallel to itself from point A to point B, then its direction at B will be independent of its path – this is examined more closely in Chapter 3. This flatness is a property of the metric, and is not a property of a particular choice of coordinates; Euclidean space is just as flat in polar coordinates as it is in cartesian coordinates.

The difference between euclidean and Minkowski space is that, for any choice of coordinates in a euclidean space, we can, with suitable amounts of linear algebra, find a coordinate transformation that changes the metric in those coordinates (such as Eq. (2.32)) into the cartesian metric of Eq. (2.26), and, for

any coordinates in Minkowski space, we can change to coordinates in which the metric is Eq. (2.33).

The basis vectors $\{\bar{e}_x, \bar{e}_y, \ldots\}$, in which the metric is Eq. (2.26), are a *cartesian basis*, and the basis vectors defined just above Eq. (2.33) are a *Lorentz basis*; in each case, a constant vector field will have the same components at all points in the space (we will come back to this in Chapter 3; the fact that we can do this at any point of a curved space, to get 'locally inertial' coordinates, will be crucial in Section 3.3.1).

Exercises

Exercises 1–12 of Schutz's chapter 3 are also useful.

Exercise 2.1 (§ 2.1.1) Demonstrate that the set of 'ordinary vectors' does indeed satisfy these axioms. Demonstrate that the set of all functions also satisfies them and is thus a vector space. Demonstrate that the subset of functions $\{e^{ax} : a \in \mathbb{R}\}$ is a vector space (hint: think about what the 'vector addition' operator should be in this case). Can you think of other examples? $[u^+]$

Exercise 2.2 (§ 2.1.1) Prove that if $\{b_i\}$ are the components of an arbitrary vector B with respect to an orthonormal basis $\{e_i\}$, then $b_i = B \cdot e_i$. $[d^-]$

Exercise 2.3 (§ 2.2.5) Tensor components A^{ij} and B^{ij} are equal in one coordinate frame. By considering the transformation law for a $\binom{2}{0}$ tensor (introduced later in this part, in Section 2.2.7), show that they must be equal in any coordinate frame. Show that if A^{ij} is symmetric in one coordinate frame, it is symmetric in any frame. $[d^- u^+]$

Exercise 2.4 (§ 2.2.5) So, why are Eq. (2.8) and Eq. (2.9) obvious? $[d^- u^+]$

Exercise 2.5 (§ 2.2.5) Show that the contraction in Eq. (2.11) is indeed a tensor. You will need to show that S is linear in its arguments, and independent of the choice of basis vectors $\{\bar{e}_i\}$ (you will need Section 2.2.7).

Exercise 2.6 (§ 2.2.6) Justify each of the steps in Eq. (2.14). $[d^- u^+]$

Exercise 2.7 (§ 2.2.6) (a) Given that T is a $\binom{1}{2}$ tensor, $\overline{A}$ and $\overline{B}$ are vectors, $\widetilde{p}$ is a one-form, and g is the metric, give the $\binom{M}{N}$ rank of each of the following

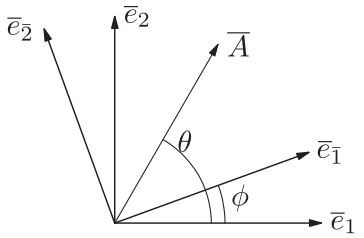

Figure 2.6 Two coordinate systems.

objects, where as usual $\cdot$ represents an unfilled argument to the function T (not all the following are valid; if not, say why):

1. $\overline{A}(\widetilde{\cdot})$ 2. $\widetilde{p}(\overline{\cdot})$ 3. $\mathsf{T}(\widetilde{\cdot}, \overline{\cdot}, \overline{\cdot})$ 4. $\mathsf{T}(\widetilde{p}, \overline{\cdot}, \overline{\cdot})$
5. $\mathsf{T}(\widetilde{p}, \overline{A}, \overline{\cdot})$ 6. $\mathsf{T}(\widetilde{\cdot}, \overline{A}, \overline{\cdot})$ 7. $\mathsf{T}(\widetilde{\cdot}, \overline{\cdot}, \overline{B})$ 8. $\mathsf{T}(\widetilde{\cdot}, \overline{A}, \overline{B})$
9. $\mathsf{T}(\widetilde{p}, \overline{A}, \overline{B})$ 10. $\overline{A}(\overline{\cdot})$ 11. $\widetilde{p}(\widetilde{\cdot})$

(b) State which of the following are valid expressions representing the components of a tensor. For each of the expressions that is a tensor, state the $\binom{M}{N}$ type of the tensor; and for each expression that is not, explain why not.

1. g_{ii} 2. $g_{ij}T^{j}{}_{kl}$ 3. $g_{ik}T^{j}{}_{kl}$ 4. $T^{i}{}_{ij}$
5. $g_{ij}A^{i}A^{j}$ 6. $g_{ij}A^{k}A^{k}$

If you're looking at this after studying Chapter 3, then how about (7) $A^{i}{}_{,j}$ and (8) $A^{i}{}_{;j}$?

What about $A^{i} = \overline{A}(\widetilde{\omega}^{i})$? [$d^{-}$]

Exercise 2.8 (§ 2.2.7) Figure 2.6 shows a vector $\overline{A}$ in two different coordinate systems. We have $\overline{A} = \cos\theta\overline{e}_1 + \sin\theta\overline{e}_2 = \cos(\theta - \phi)\overline{e}_{\bar{1}} + \sin(\theta - \phi)\overline{e}_{\bar{2}}$. Obtain $\overline{e}_{1,2}$ in terms of $\overline{e}_{\bar{1},\bar{2}}$ and vice versa, and obtain $A^{\bar{i}}$ in terms of $\cos\theta$, $\cos\phi$, $\sin\theta$ and $\sin\phi$. Thus identify the components of the matrix $\Lambda^{i}_{\bar{i}}$ (see also Eq. (2.23)).

Exercise 2.9 (§ 2.2.7) Show that the tensor S in Eq. (2.10) is independent of the basis vectors used to define it.

Exercise 2.10 (§ 2.2.7) Make a table showing all the transformations $\overline{e}_i \leftrightarrow \overline{e}_{\bar{i}}$, $A^{i} \leftrightarrow A^{\bar{i}}$, $\widetilde{\omega}^{i} \leftrightarrow \widetilde{\omega}^{\bar{i}}$ and $p_i \leftrightarrow p_{\bar{i}}$, patterned after Eq. (2.17) and Section 3.1.4. You will need to use the fact that the Λ matrices are inverses of each other, and that $\overline{A} = A^{i}\overline{e}_i = A^{\bar{i}}\overline{e}_{\bar{i}}$. This is a repetitive, slightly tedious, but extremely valuable exercise. One point of this is to emphasise that once you have chosen the transformation for one of these (say, $\overline{e}_{\bar{i}} = \Lambda^{i}_{\bar{i}}\overline{e}_i$), all of the others are determined. [$d^{-}u^{++}$]

Exercise 2.11 (§ 2.2.7) By repeating the logic that led to Eq. (2.17), prove Eq. (2.19) and Eq. (2.24) [use $\mathsf{T} = T^{ij}{}_k(\bar{e}_i \otimes \bar{e}_j \otimes \tilde{\omega}^k)$]. Alternatively, and more directly, use the results of Exercise 2.10. $[d^+]$

Exercise 2.12 (§ 2.3.2) Consider the vector $\overline{A} = 1\bar{e}_1 + 1\bar{e}_2$, where $\bar{e}_1$ and $\bar{e}_2$ are the cartesian basis vectors; observe that this vector has the same components at all points in $\mathbb{R}^2$. Using Eq. (2.17) and the appropriate transformation matrix for polar coordinates, determine the components of this vector in the polar basis, as evaluated at the points $(r, \theta) = (1, \pi/4)$, $(1, 0)$, $(2, \pi/4)$ and $(2, 0)$? Use the metric for these coordinates, Eq. (2.32), to find the length of $\overline{A}$ at each of these points. What happens at the origin? $[u^+]$

Exercise 2.13 (§ 2.3.4) (i) Given a vector with components A^μ in a Minkowski space, write down its coordinates in a different set of Minkowski coordinates, obtained from the first by Eq. (2.34), and verify that these match the expression for Lorentz-transformed 4-vectors, which you learned about in SR.

(ii) What is the inverse, $\Lambda^{\mu}_{\bar{\mu}}$ of Eq. (2.34)? Use it to write down the components of the metric tensor Eq. (2.33) after transformation by this Λ, and verify that $g_{\bar{\mu}\bar{\nu}} = \eta_{\bar{\mu}\bar{\nu}}$.

(iii) Consider the transformation matrix

$$\Lambda^{\mu}_{\bar{\mu}} = \begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{i}$$

which is the simplest transformation that 'mixes' the x- and t-coordinates. By *requiring* that $g_{\bar{\mu}\bar{\nu}} = \eta_{\bar{\mu}\bar{\nu}}$ after transformation by this Λ, find constraints on the parameters a, b, c, and d (you can freely add the constraint $b = c$; why?), and so deduce the matrix Eq. (2.34). $[u^+]$

3

Manifolds, Vectors, and Differentiation

In the previous chapter, we carefully worked out the various things we can do with a set of vectors, one-forms, and tensors, once we have identified those objects. Identifying those objects on a curved surface is precisely what we are about to do now. We discover that we have to take a rather roundabout route to them.

After defining them suitably, we next want to differentiate them. But as soon as we can do that, we have all the mathematical technology we need to return to physics in the next chapter, and start describing gravity.

3.1 The Tangent Vector

You learned about vectors first in school, perhaps by visualising them as 'a thing which points', and possibly defining them using the separations between points, or using differences in coordinate values, or otherwise defining them using their components in some coordinate system. That is straightforward when using euclidean coordinates, but becomes rapidly challenging when doing so using a general coordinate system. One can develop GR using this broad approach – Rindler (2006) does so, for example – but it is messy, is arguably more confusing, and is now regarded as rather old-fashioned. MTW (1973) were amongst the first to popularise a more 'geometric' or 'coordinate-free' approach that stresses the basis-independence of physical quantities, but which requires a more subtle approach to defining vectors. This approach defines vectors at a point, with no remnant of the separations between coordinate points that we learned at first.

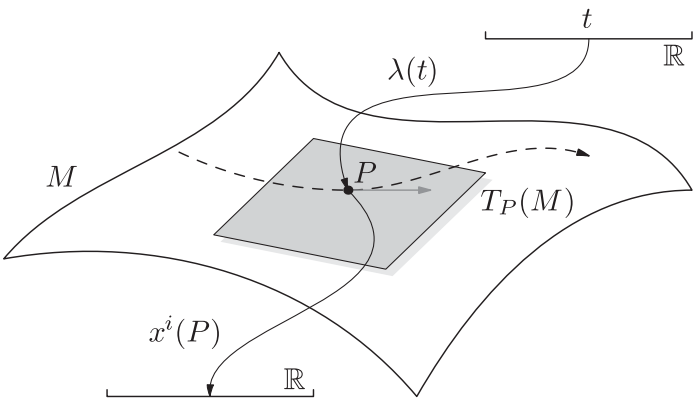

Figure 3.1 A manifold M, with a curve $\lambda : \mathbb{R} \to M$ (dashed), a point on the manifold $P = \lambda(t)$, and a coordinate function $x^i : M \to \mathbb{R}$. Curves that go through the point P have tangent vectors, at P (shaded arrow), which lie in the space $T_P(M)$ (shaded). The function $x^i(t) = (x^i \circ \lambda)(t)$ is therefore $\mathbb{R} \to \mathbb{R}$.

3.1.1 Manifolds and Functions

The arena in which everything happens is the *manifold* . A manifold is a set of points, with the only extra structure being enough to allow continuous functions to be defined on it (Figure 3.1). In particular, a manifold does not have a metric defined.

A *chart* is a set of *functions* $\{x^1, \ldots, x^n\}$ that between them map points on the manifold to $\mathbb{R}^n$. We may need more than one chart to cover all of the manifold. In other words, it is a *coordinate system* , $x^i : M \to \mathbb{R}$. The fact that the range of this map is (flat) $\mathbb{R}^n$ allows us to say that the manifold is *locally euclidean* .

Now consider a *path* on the manifold – this is just a continuous sequence of points. We distinguish this from a *curve* , $\lambda(t) : \mathbb{R} \to M$, which is a mapping from a parameter t to points on a path – two mappings that map to the same path but with different parameterisation are different curves.

If we put these ideas together, and think of the functions $x^1\big(\lambda(t)\big), \ldots,$ $x^n\big(\lambda(t)\big)$, then we have a set of mappings from the curve parameter to the coordinates. The properties of the manifold tell us that these are smooth functions $x^i(t) = x^i(\lambda(t)) : \mathbb{R} \to \mathbb{R}$, so we can differentiate with respect to the parameter, t.

We can regard a chart as a *reference frame* , and I will generally use the latter term below.

There's quite a lot we *could* say about manifolds: for a mathematician there's a long backstory before we get to this starting point. Mathematicians have to be precise about what minimal structures must be present before a definition of differentiation is possible; in other words, what

structures are there, the removal of which would make it impossible to define a derivative? For our present purposes we can do the traditional physicists' thing and ignore such niceties, and assume that our spaces of interest are well enough behaved that they can support coordinates and curves as discussed in this section. Later in your study of GR, you may have to become aware of these conditions again, in a detailed study of black holes (the point of singularity is not in the manifold, or more specifically in any open subset of it) or the large-scale structure of space-time (where the overall topology of the space becomes important).

That said, for completeness, I'll mention a few details about what structure we already have at this point. A manifold, M, is a set (which we naturally think of here as a set of points) in which we can identify open subsets; we can identify such a subset (that is, a 'neighbourhood') for every point in the manifold ($\forall p \in M, \exists S \subset M$ such that $p \in S$). We can smoothly patch together multiple such subsets to cover the manifold, even though there might not be any single subset that covers the entire manifold. At this point we have a 'topological space'. We then may or may not be able to define maps from each of these subsets to $\mathbb{R}^n$ (these are the aforementioned 'charts'); if we can, and with the same n in all cases, we have an n-dimensional manifold (that is, each such subset is homeomorphic to $\mathbb{R}^n$; two spaces are homeomorphic if there is a continuous bijection with a continuous inverse, which maps one to the other; each such subset 'looks like' $\mathbb{R}^n$). That is, the dimension of the manifold, n, is a property of the manifold, and not a consequence of any arbitrariness in the number of coordinate functions we use. These maps must be continuous, but we will also assume that they are as differentiable as we need them to be. Carroll (2004, § 2.2) has an excellent description of the sequence of ideas, along with examples of spaces that are and are not manifolds. It's also possible to say quite a lot more about the precise relationship between charts, coordinates, and frames, but we already have as much detail as we need.

3.1.2 Defining the Tangent Vector

Now think of a function $f : M \rightarrow \mathbb{R}$ which is defined on the manifold M, and therefore at every point along the curve λ. The function f is therefore also a function ($\mathbb{R}^n \rightarrow \mathbb{R}$) of the coordinates of the points along that curve, or

$$f = f\big(\lambda(t)\big) = f\Big(x^1\big(\lambda(t)\big), \ldots, x^n\big(\lambda(t)\big)\Big),$$

which we can write as just ($\mathbb{R} \rightarrow \mathbb{R}$)

$$f = f\big(x^1(t), \ldots, x^n(t)\big).$$

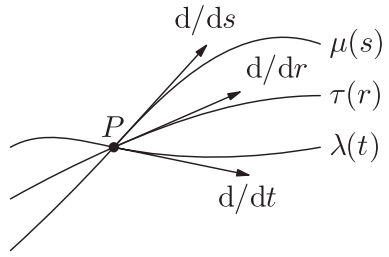

Figure 3.2 Multiple curves through P.

Be aware that, strictly, we are talking about three different functions here, namely $f(P) : M \to \mathbb{R}$, and $f(x^1, \ldots, x^n) : \mathbb{R}^n \to \mathbb{R}$, and $f(x^1(t), \ldots, x^n(t)) : \mathbb{R} \to \mathbb{R}$. Giving all three functions the same name, f, is a sort of pun. Similarly, we will think of x^i as interchangeably a function $x^i(P) : M \to \mathbb{R}$, or $x^i(\lambda(t)) : \mathbb{R} \to \mathbb{R}$, or as the number that is one of the arguments of the function $f(x^1, \ldots, x^n)$. When we manipulate these objects in the rest of the book, it should be clear which interpretation we mean: when we write $\partial f/\partial x^i$, for example, we are thinking of f as $f(x^1, \ldots, x^n)$, and thinking of the x^i as numbers; when we write $\partial x^i/\partial x^j = \delta^i{}_j$ we simply mean that the coordinate function x^i is independent of the value of the coordinate x^j.

So how does f vary as we move along the curve? Easy:

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \sum_{i=1}^{n} \frac{\partial x^i}{\partial t} \frac{\partial f}{\partial x^i}.$$

However, since this is true of *any* function f, we can write instead

$$\frac{\mathrm{d}}{\mathrm{d}t} = \sum_i \frac{\partial x^i}{\partial t} \frac{\partial}{\partial x^i}. \tag{3.1}$$

We can now derive two important properties of this derivative. Consider the same path parameterised by $t_a = t/a$. We have

$$\begin{aligned} \frac{\mathrm{d}f}{\mathrm{d}t_a} &= \sum_i \frac{\partial x^i}{\partial t_a} \frac{\partial f}{\partial x^i} \\ &= a \sum_i \frac{\partial x^i}{\partial t} \frac{\partial f}{\partial x^i} \\ &= a \frac{\mathrm{d}f}{\mathrm{d}t} \end{aligned} \tag{3.2}$$

Next, consider another curve $\mu(s)$, which crosses curve $\lambda(t)$ at point P. We can therefore write, at P,

$$a \frac{\mathrm{d}f}{\mathrm{d}s} + b \frac{\mathrm{d}f}{\mathrm{d}t} = \sum_i \left(a \frac{\partial x^i}{\partial s} + b \frac{\partial x^i}{\partial t} \right) \frac{\partial f}{\partial x^i} = \sum_i \frac{\partial x^i}{\partial r} \frac{\partial f}{\partial x^i} = \frac{\mathrm{d}f}{\mathrm{d}r}, \tag{3.3}$$

for some further curve $\tau(r)$ that also passes through point P (see Figure 3.2).

But now look what we have discovered. Whatever sort of thing $\mathrm{d}/\mathrm{d}t$ is, $a\,\mathrm{d}/\mathrm{d}t$ is the same type of thing (from Eq. (3.2)), and so is $a\,\mathrm{d}/\mathrm{d}s + b\,\mathrm{d}/\mathrm{d}t$. But now we can look at Section 2.1.1, and realise that these derivative-things *defined at* P, which we'll write $(\mathrm{d}/\mathrm{d}t)_P$, satisfy the axioms of a *vector space* . Thus the things $(\mathrm{d}/\mathrm{d}t)_P$ are another example of things that can be regarded as vectors, or $\binom{1}{0}$ tensors. The thing $(\mathrm{d}/\mathrm{d}t)_P$ is referred to as a *tangent vector* .

When we talk of 'vectors' from here on, it is these tangent vectors that we mean.

A vector $\overline{V} = (\mathrm{d}/\mathrm{d}t)_P$ has rather a double life. Viewed as a derivative, $\overline{V}$ is just an operator, which acts on a function f to give

$$\overline{V}f = \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_P f = \left.\frac{\mathrm{d}f}{\mathrm{d}t}\right|_{t(P)},$$

the rate of change of f along the curve $\lambda(t)$, evaluated at P. There's nothing particularly exotic there. What we have just discovered, however, is that this object $\overline{V} = (\mathrm{d}/\mathrm{d}t)_P$ can *also, separately* , be regarded as an element of a vector space associated with the point P, and as such is a $\binom{1}{0}$ tensor, which is to say a thing that takes a *one-form* as an argument, to produce a number that we will write as $\langle\widetilde{\omega}, \overline{V}\rangle$, for some one-form $\widetilde{\omega}$ (we will see in a moment what this one-form is; it is not the function f). This dual aspect does seem confusing, and makes the object $\overline{V}$ seem more exotic than it really is, but it will (or should be!) always clear from context which facet of the vector is being referred to at any point.

We'll denote the set of these directional derivatives as $T_P(M)$, the *tangent plane of the manifold* M *at the point* P. It is very important to note that $T_P(M)$ and, say, $T_Q(M)$ – the tangent planes at two different points, P and Q, of the manifold – are *different spaces* , and have nothing to do with one another a priori (though we want them to be related, and this is ultimately why we introduce the connection in Section 3.2).

With this in mind, we can reread Eq. (3.1) as a vector equation, identifying the vectors

$$\overline{e}_i = \left(\frac{\partial}{\partial x^i}\right)_P \tag{3.4}$$

as a basis for the tangent-plane, and the *numbers* $\partial x^i/\partial t$ as the components of the vector $\overline{V} = (\mathrm{d}/\mathrm{d}t)_P$ in this basis, or

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_P = \sum_i \frac{\partial x^i}{\partial t}\left(\frac{\partial}{\partial x^i}\right)_P$$
$$\overline{V} = V^i \overline{e}_i.$$

So, I've shown you that we can regard the $(\mathrm{d}/\mathrm{d}t)_P$ as vectors; the rest of this part of the book should convince you that this is additionally a useful thing to do.

3.1.3 The Gradient One-Form

Consider a function f, defined on the manifold. This is a *field*, which is to say it is a rule that associates an object – in this case the number that is the value of the function – with each point on the manifold (see Section 2.2.3). Given this function, there is a particular one-form field that we can define (that is, a rule for associating a one-form with each point in the manifold), namely the *gradient one-form* $\widetilde{\mathrm{d}}f$. Given a vector $\overline{V} = (\mathrm{d}/\mathrm{d}t)_P$, which is the tangent to a curve $\lambda(t)$, the gradient one-form is *defined* by its contraction with this vector:

$$\left\langle \widetilde{\mathrm{d}}f, \overline{V} \right\rangle = \left\langle \widetilde{\mathrm{d}}f, \frac{\mathrm{d}}{\mathrm{d}t} \right\rangle = \widetilde{\mathrm{d}}f \left(\frac{\mathrm{d}}{\mathrm{d}t} \right) \equiv \left. \frac{\mathrm{d}f}{\mathrm{d}t} \right|_P . \tag{3.5}$$

The first two equalities here simply express notational equivalences; it is the third equivalence that constitutes the definition of the gradient one-form's action. This contraction between a vector and a gradient field is illustrated in Figure 2.3.

What does Eq. (3.5) look like in component form? Writing $\overline{V} = \mathrm{d}/\mathrm{d}t = V^i \overline{e}_i$, we can write

$$\left. \frac{\mathrm{d}f}{\mathrm{d}t} \right|_P = \left\langle \widetilde{\mathrm{d}}f, \overline{V} \right\rangle = V^i \left\langle \widetilde{\mathrm{d}}f, \overline{e}_i \right\rangle = V^i \frac{\mathrm{d}f}{\mathrm{d}x^i}. \tag{3.6}$$

Now consider the gradient one-form associated with, not f, but one of the coordinate functions x^i (from Section 3.1.1, recall that the coordinates are just a set of functions on the manifold, and in this sense not importantly different from an arbitrary function f). We write these one-forms as simply $\widetilde{\mathrm{d}}x^i$: what is their action on the basis vectors $\overline{e}_i = \partial/\partial x^i$ (from Eq. (3.4))? Directly from Eq. (3.5),

$$\widetilde{\mathrm{d}}x^i \left(\frac{\partial}{\partial x^j} \right) = \frac{\partial x^i}{\partial x^j} = \delta^i{}_j \tag{3.7}$$

so that, comparing this with Eq. (2.5), we see that the set $\widetilde{\omega}^i = \widetilde{\mathrm{d}}x^i$ forms a basis for the one-forms, which is *dual* to the vector basis $\overline{e}_i = \partial/\partial x^i$.

[Exercises 3.1 and 3.2]

3.1.4 Basis Transformations

What does a change of basis look like in this new notation? If we decide that we do not like the coordinate functions x^i and decide to use instead

functions $x^{\bar{\imath}}$, how does this appear in our formalism, and how does it compare to Section 2.2.7?

The new coordinates will generate a set of basis vectors

$$\overline{e}_{\bar{\imath}} = \frac{\partial}{\partial x^{\bar{\imath}}}. \tag{3.8}$$

This new basis will be related to the old one by a linear transformation

$$\overline{e}_{\bar{\imath}} = \Lambda^{j}_{\bar{\imath}} \overline{e}_j$$

and the corresponding one-form basis will be related via the inverse transformation

$$\widetilde{\omega}^{\bar{\imath}} = \Lambda^{\bar{\imath}}_{j} \widetilde{\omega}^{j}$$

(recall Exercise 2.10). Thus, from Eq. (2.18),

$$\Lambda^{\bar{\imath}}_{j} = \widetilde{\omega}^{\bar{\imath}}(\overline{e}_j) = \widetilde{\mathrm{d}}x^{\bar{\imath}} \left(\frac{\partial}{\partial x^j} \right) = \frac{\partial x^{\bar{\imath}}}{\partial x^j} \tag{3.9a}$$

$$\Lambda^{j}_{\bar{\imath}} = \widetilde{\omega}^{j}(\overline{e}_{\bar{\imath}}) = \widetilde{\mathrm{d}}x^{j} \left(\frac{\partial}{\partial x^{\bar{\imath}}} \right) = \frac{\partial x^{j}}{\partial x^{\bar{\imath}}}. \tag{3.9b}$$

Note that Eq. (3.8) does the right thing if we, for example, double the value of the coordinate function x^i. If x^i doubles, then $\partial f/\partial x^i$ halves, but Eq. (3.8) then implies that $\overline{e}_i$ halves, which means that the corresponding component of $\overline{V}$, namely V^i, doubles, so that $V^i \partial f/\partial x^i$ is unchanged, as expected.

Note that the transformation matrix is defined as transforming the basis vectors and one-forms; as an immediate consequence it can transform the vector and one-form components also (as discussed in Section 2.2.7). Because of the *choice* in Eq. (3.8) of basis vectors as the differentials of the coordinate functions, the transformation matrix *also* describes a transformation between coordinate systems. This choice of basis vectors is a *coordinate basis* – see the 'dangerous bend' paragraph Section 2.3.2 for discussion of non-coordinate bases.

If we consider a curve $\lambda(t)$, which is such that $\partial x^1\big(\lambda(t)\big)/\partial t = 1$ and $x^i\big(\lambda(t)\big) = c^i$ (constant) for $i > 1$ (i.e., this is a 'grid line'), then simply comparing with Eq. (3.1) we see that $\mathrm{d}/\mathrm{d}t = \partial/\partial x^1$. Thus in a coordinate basis, where $\overline{e}_i = \partial/\partial x^i$, the ith basis vector at any point is tangent to the ith 'grid line', which matches the intuition we have for the basis vectors $\overline{e}_x$, $\overline{e}_y$, and so on, in ordinary plane geometry.

[Exercises 3.3–3.5]

3.2 Covariant Differentiation in Flat Spaces

We are now finally in a position to move on to the central tool of this chapter, the ideas of coordinate-independent differentiation of tensors, parallel transport, and curvature. We will make this move in two steps: first, we will learn how to handle the situation where the basis vectors of the space of interest are different at different points in the space, but confining ourselves to flat (euclidean) space, where we already know how to do most of the calculations; second, we will discover the rather simple step involved in transferring this knowledge to the case of fully curved spaces.

There are other ways of introducing the covariant derivative, which are very insightful, but more than a little abstract. Stewart (1991, §1.7) introduces it in an axiomatic way which makes clear the tensorial nature of the covariant derivative from the very outset, as well as its linearities and some of its other properties. Schutz (1980, chapter 6) introduces it in a typically elegant way, via parallel transport, and emphasising the ultimate arbitrariness of the precise differentiation rule. Both of these routes define a connection that is more general than the one that, following Schutz (i.e., Schutz (2009)), we are about to derive, which is known as the 'metric connection'. Both of these books promptly specialise to the metric connection, but it seems useful here to build up this specific connection step by step, emphasising the link to changes of bases, and going via the covariant derivative in flat spaces. Equation (3.35), which we derive rather than assert, is the defining property of this connection. Chapter 10 of MTW gives a very good, and visual, introduction to covariant differentiation, though approaching it from a somewhat different direction.

The point of this tool – the goal we are aiming for – is this: given some geometrical object $\overline{V}$ of physical interest (such as an electric field in a space, or a strain tensor in some medium), we want to be able to talk about how it varies as we move around a space, in a way that doesn't depend on the coordinates we have chosen.

3.2.1 Differentiation of Basis Vectors

This section is to some extent another notation section, in that it is describing something you already know how to do, but in more elaborate and powerful language.

You will in the past have dealt with calculus in curvilinear coordinate systems and produced such results as the Laplacian in spherical polar coordinates

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}.$$

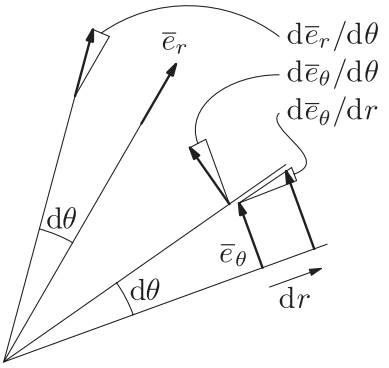

Figure 3.3 The basis of polar coordinates, and their derivatives.

We are now aiming for much the same destination, but by a slightly different route. This follows Schutz §§5.3–5.5 quite closely.

In order to illustrate the process, we will examine the basis vectors of (plane) polar coordinates, as expressed in terms of the cartesian basis vectors $\overline{e}_x$ and $\overline{e}_y$. We will promptly see that our formalism is not restricted to this route.

The basis vectors of polar coordinates are

$$\overline{e}_r = \cos\theta\overline{e}_x + \sin\theta\overline{e}_y \tag{3.10a}$$

$$\overline{e}_\theta = -r\sin\theta\overline{e}_x + r\cos\theta\overline{e}_y \tag{3.10b}$$

(compare the 'dangerous bend' discussion of coordinate bases in Section 2.3.2). A little algebra shows that

$$\frac{\partial}{\partial r}\overline{e}_r = 0 \tag{3.11a}$$

$$\frac{\partial}{\partial\theta}\overline{e}_r = \frac{1}{r}\overline{e}_\theta \tag{3.11b}$$

$$\frac{\partial}{\partial r}\overline{e}_\theta = \frac{1}{r}\overline{e}_\theta \tag{3.11c}$$

$$\frac{\partial}{\partial\theta}\overline{e}_\theta = -r\overline{e}_r \tag{3.11d}$$

so that we can see how the basis vectors *change* as we move to different points in the plane (Figure 3.3), unlike the cartesian basis vectors.

At any point in the plane, a vector $\overline{V}$ has components (V^r, V^θ) in the polar basis *at that point*. We can differentiate this vector with respect to, say, r, in the obvious way

$$\begin{aligned}\frac{\partial\overline{V}}{\partial r} &= \frac{\partial}{\partial r}(V^r\overline{e}_r + V^\theta\overline{e}_\theta)\\ &= \frac{\partial V^r}{\partial r}\overline{e}_r + V^r\frac{\partial\overline{e}_r}{\partial r} + \frac{\partial V^\theta}{\partial r}\overline{e}_\theta + V^\theta\frac{\partial\overline{e}_\theta}{\partial r},\end{aligned}$$

or, in index notation, with the summation index i running over the 'indexes' r and θ,

$$\begin{aligned}\frac{\partial \overline{V}}{\partial r} &= \frac{\partial}{\partial r}(V^i \overline{e}_i) \\ &= \frac{\partial V^i}{\partial r}\overline{e}_i + V^i \frac{\partial \overline{e}_i}{\partial r}.\end{aligned} \tag{3.12}$$

So much for polar coordinates.

Having illustrated the process with polar coordinates, we can obtain the general expression either by replacing r, in Eq. (3.12), by a general coordinate x^j, or (more directly) by using the Leibniz rule on $\overline{V} = V^i \overline{e}_i$, to obtain

$$\frac{\partial \overline{V}}{\partial x^j} = \frac{\partial V^i}{\partial x^j}\overline{e}_i + V^i \frac{\partial \overline{e}_i}{\partial x^j}. \tag{3.13}$$

In cartesian coordinates, the second term in this expression is identically zero, since the basis vectors are the same everywhere on the plane, and so in those coordinates we can obtain the derivative of a vector by simply differentiating its components (the first term in Eq. (3.13)). This is not true when we are using curvilinear coordinates, and the second term comes in when we are obliged to worry about how the basis vectors are different at different points on the plane.

Now, the second term in Eq. (3.13), $\partial \overline{e}_i / \partial x^j$, is itself a vector, so that it is a linear combination of the basis vectors, with coefficients Γ^k_{ij}:

$$\frac{\partial \overline{e}_i}{\partial x^j} = \Gamma^k_{ij} \overline{e}_k. \tag{3.14}$$

These coefficients Γ^k_{ij} are known as the *Christoffel symbols*, and this set of $n \times n \times n$ numbers encodes all the information we need about how the coordinates, and their associated basis vectors, change within the space.[1] The object Γ is not a tensor – it is merely a collection of numbers – so its indexes are not staggered (just like the transformation matrix Λ).

Returning to polar coordinates, we discover that we have already done all the work required to calculate the relevant Christoffel symbols. If we compare Eq. (3.14) with Eq. (3.11) (replacing $\overline{e}_r \mapsto \overline{e}_1$ and $\overline{e}_\theta \mapsto \overline{e}_2$), we see, for example, that

$$\begin{aligned}\frac{\partial \overline{e}_1}{\partial x^2} = \frac{\partial \overline{e}_r}{\partial \theta} &= \Gamma^1_{12}\overline{e}_1 + \Gamma^2_{12}\overline{e}_2 \\ &= 0\overline{e}_1 + \frac{1}{r}\overline{e}_2,\end{aligned}$$

[1] There seems to be no universal consensus on whether Γ^i_{jk} is referred to as the Christoffel 'symbol' or 'symbols', plural. The former seems more rational, but the latter seems less awkward in prose, and is the version that I will generally use in the text to follow.

so that $\Gamma^1_{12} = 0$ and $\Gamma^2_{12} = 1/r$. We will sometimes write this, slightly slangily, as $\Gamma^r_{r\theta} = 0$ and $\Gamma^\theta_{r\theta} = 1/r$. By continuing to match Eq. (3.11) with Eq. (3.14), we find

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r}, \qquad \Gamma^1_{22} = -r, \qquad \text{others zero,} \tag{3.15}$$

or

$$\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{1}{r}, \qquad \Gamma^r_{\theta\theta} = -r, \qquad \text{others zero.} \tag{3.16}$$

[Exercise 3.6]

3.2.2 The Covariant Derivative in Flat Spaces

More notation: If we rewrite Eq. (3.13) including Eq. (3.14), relabel and reorder, we find

$$\frac{\partial \overline{V}}{\partial x^j} = \left(\frac{\partial V^i}{\partial x^j} + V^k \Gamma^i_{kj} \right) \overline{e}_i. \tag{3.17}$$

For each j this is a vector at each point in the space – that is to say, it is a vector field – with components given by the term in brackets. We denote these components of the vector field by the notation $V^i{}_{;j}$, where the semicolon denotes covariant differentiation. We can further denote the derivative of the component with a comma: $\partial V^i/\partial x^j \equiv V^i{}_{,j}$. Then we can write

$$\frac{\partial \overline{V}}{\partial x^j} = V^i{}_{;j} \overline{e}_i \tag{3.18a}$$

$$V^i{}_{;j} = V^i{}_{,j} + V^k \Gamma^i_{kj}. \tag{3.18b}$$

It is important to be clear about what you are looking at, here. The objects $V^i{}_{;j}$ are numbers, which are the *components* , indexed by i, of a *set of vectors*, indexed by j. They look rather like tensor components, however, because of how we have chosen to write them; and we are about to deduce that that is exactly what they are in fact.[2] But components of which tensor?

Final step: looking back at Eq. (3.8), we see that the differential operator $\partial/\partial x^j$ in Eq. (3.17) is associated with the basis vector $\overline{e}_j$, and this is consistent with what we saw in Section 3.1.4: that $\overline{e}_j$ is proportional to $\partial/\partial x^j$, and thus that $\partial \overline{V}/\partial x^j$, in Eq. (3.18a), is proportional to $\overline{e}_j$ also. That linearity permits us to define a $\binom{1}{1}$ tensor, which we shall call $\nabla \overline{V}$, which we shall *define* by saying

[2] It's also unfortunate that the notation includes common punctuation characters: at the risk of stating the obvious, note that any commas or semicolons *following* such notation is part of the surrounding text.

that the action of it on the vector $\overline{e}_j$ is the vector $\partial\overline{V}/\partial x^j$ in Eq. (3.17). That is, using the notation of Chapter 2, we could write

$$(\nabla\,\overline{V})(\tilde{\cdot}\,,\overline{e}_j) \equiv \frac{\partial\overline{V}}{\partial x^j}(\tilde{\cdot}\,) \tag{3.19}$$

as the definition of the tensor $\nabla\,\overline{V}$. For notational convenience, we prefer to write this instead as

$$\nabla_{\overline{e}_j}\overline{V} = \frac{\partial\overline{V}}{\partial x^j}, \tag{3.20}$$

where both sides of this equation are, of course, vectors.

This tensor $\nabla\,\overline{V}$ is called the *covariant derivative of* $\overline{V}$, and its components are

$$(\nabla\,\overline{V})^i{}_j \equiv (\nabla\,\overline{V})(\tilde{\omega}^i;\overline{e}_j) \equiv (\nabla_{\overline{e}_j}\overline{V})^i \equiv (\nabla_j\overline{V})^i = V^i{}_{;j}, \tag{3.21}$$

where the first equivalence is what we mean by the components of a tensor, the second is the definition of the tensor, restated from the text immediately above Eq. (3.19), the third is a notational convenience, which applies in the case where the argument vector is a basis vector, and the equality indicates the numerical value of this object – the ith component of the vector $\nabla_j\overline{V}$ – via Eq. (3.20) and Eq. (3.18a).

You will also sometimes see an expression such as $\nabla_{\overline{X}}\overline{V}$. This is the covariant derivative of $\overline{V}$, contracted with $\overline{X}$. In component form, this is

$$\nabla_{\overline{X}}\overline{V} = \nabla\,\overline{V}(\tilde{\cdot}\,,\overline{X}) = X^i\nabla\,\overline{V}(\tilde{\cdot}\,,\overline{e}_i) = X^i\nabla_i\overline{V} = X^iV^j{}_{;i}\overline{e}_j. \tag{3.22}$$

We have introduced a blizzard of notations here. Remember that they are all notational variants of the same underlying object, namely the tensor $\nabla\,\overline{V}$. Make sure you understand how to go from one variant to the other, and why they relate in the way they do.

Note : it is easy to misread the notation $\nabla_{\overline{X}}\overline{Y}$ as being some sort of tensorial operation of ∇ on arguments $\overline{X}$ and $\overline{Y}$, and thus to leap to the conclusion that this is therefore linear in $\overline{X}$ and $\overline{Y}$. That would be wrong. We have here a $\binom{1}{0}$ tensor $\overline{Y}$ and a corresponding $\binom{1}{1}$ tensor $\nabla\,\overline{Y}$. If we supply the latter with two arguments $\overline{X}$ and $\tilde{p}$ then we could write the result as $\nabla\,\overline{Y}(\tilde{p},\overline{X})$ or, looking at the first notational eccentricity of Eq. (3.22) as $\nabla_{\overline{X}}\overline{Y}(\tilde{p})$. That reminds us that the thing written $\nabla_{\overline{X}}\overline{Y}$ is a vector. Exercise 3.10 is instructive here.

Here's where we've got to: we've managed to define a *tensor* field related to $\overline{V}$, called the covariant derivative, and written $\nabla\,\overline{V}$, which (since it is a tensor) is independent of any coordinate system, and so, in particular, doesn't pick out any coordinate system as special. If we need its components in a specific system $\{x^k\}$, however, because we need to do some calculations, we

can find them easily, via Eq. (3.18), or by transforming the components from a system where we already know them (such as cartesian coordinates) into the system $\{x^k\}$ – we know we can do this because we know that $\nabla \overline{V}$ is a tensor, so we know how its components transform.

Finally, here, note that a scalar is independent of any coordinate system, therefore all the complications of this section, which essentially involve dealing with the fact that basis vectors are different at different points on the manifold, disappear, and we discover that we have already obtained a covariant derivative of a scalar, in Eq. (3.5). Thus

$$\nabla f \equiv \widetilde{\mathrm{d}f} \tag{3.23}$$

and (where $\overline{V}$ is tangent to a curve with coordinate t)

$$\nabla_{\overline{V}} f = \widetilde{\mathrm{d}f}(\overline{V}) = \frac{\partial f}{\partial t}. \tag{3.24}$$

If instead we take $\overline{V} = \overline{e}_j = \partial/\partial x^j$, then

$$\nabla_j f = \frac{\partial f}{\partial x^j} \tag{3.25}$$

(cf. Schutz Eq. (5.53)). From this we can deduce the expression for the covariant derivative of a one-form, which we shall simply quote as:

$$(\nabla_j \widetilde{p})_i \equiv (\nabla \widetilde{p})_{ij} \equiv p_{i;j} = p_{i,j} - p_k \Gamma^k_{ij}. \tag{3.26}$$

Note the sign difference from Eq. (3.18).

The derivative of a $\binom{1}{1}$ tensor is

$$\nabla_j T^k{}_l \equiv T^k{}_{l;j} = T^k{}_{l,j} + \Gamma^k_{ij} T^i{}_l - \Gamma^i_{lj} T^k{}_i. \tag{3.27}$$

Note firstly how systematic this expression is, and that it is systematically extensible to tensors of higher rank – there is one $+\Gamma$ term for each upper tensor index, and one $-\Gamma$ term for each lower index. The expression looks hard to remember, but is easier than it looks, since, given the overall pattern, there is only one consistent way the indexes can fit in to each term.

Secondly, note that in Eq. (3.27) and each of the expressions that it generalises, back to Eq. (3.23), the connection ∇ acting on the tensor X forms a tensor field $\nabla \mathsf{X}$ that has one more lower index than X has – that is, it has one more vector argument than X. Supplying a vector $\overline{V}$ to this derivative produces $\nabla_{\overline{V}} \mathsf{X}$, which is the rate of change of the tensor field X along the direction $\overline{V}$.

Also, the Leibniz (or product) rule applies

$$\nabla_j (p_k V^k) = p_{k;j} V^k + p_k V^k{}_{;j}. \tag{3.28}$$

See Schutz §5.3 for details.

In this discussion of vector differentiation, built up since the beginning of Section 3.2.1, we have not had to recall anything other than that the vectors $\bar{e}_i$ are the basis vectors of a vector space. That is, there is no complication arising from our definition of the vectors as tangent vectors, associated with the derivative of a function along a curve; there is no meaningful sense in which this is a 'second derivative'.

[Exercises 3.7–3.11]

3.2.3 The Metric and the Christoffel Symbols

The covariant derivative, and the Christoffel symbols, give us information about how, and how quickly, the basis vectors change as we move about a space. It is therefore no surprise to find that there is a deep connection between these and the *metric* , which gives information about distances within a space (this argument is taken from Schutz §5.4).

Remember that the metric (a $\binom{0}{2}$ tensor) allows us to identify a particular one-form associated with a given vector:

$$\tilde{V} = \mathsf{g}(\bar{V}, \bar{\cdot}). \tag{3.29}$$

What is the derivative of this one-form?

Note that this is a purely geometrical (i.e., coordinate-independent) equation; that means that we can choose which coordinates to use, to calculate with it. We will choose cartesian coordinates, in which the basis vectors are constant, so that covariant differentiation involves the derivative of the components, only: $\nabla_j \bar{V} = \partial V^i/\partial x^j \bar{e}_i$ (this is again an application of the Leibniz rule). Thus

$$\mathsf{g}(\nabla_j \bar{V}, \bar{\cdot}) = \mathsf{g}\left(\frac{\partial V^i}{\partial x^j}\bar{e}_i, \bar{\cdot}\right) = \frac{\partial V^i}{\partial x^j}\mathsf{g}(\bar{e}_i, \bar{\cdot}). \tag{3.30}$$

If we write $\tilde{p} = \mathsf{g}(\bar{e}_i, \bar{\cdot})$, then we discover that $\tilde{p}(\bar{e}_j) = \mathsf{g}(\bar{e}_i, \bar{e}_j) = \delta_{ij}$ in these coordinates. Thus $\tilde{p}$ is one of the set of dual one-forms: $\tilde{\omega}^i = \mathsf{g}(\bar{e}_i, \bar{\cdot})$.

Differentiating now the left-hand side of Eq. (3.29) (and observing that if the basis vectors are constant, the basis one-forms must be as well), we find

$$\nabla_j \tilde{V} = \nabla_j (V_i \tilde{\omega}^i) = \frac{\partial V_i}{\partial x^j}\tilde{\omega}^i. \tag{3.31}$$

However in these coordinates, $V_i = V^i$, so that the right-hand sides of Eqs. (3.30) and (3.31) are equal, as component equations. But both of these are tensor equations, and so if the components are equal in one coordinate system, then they are equal in any coordinate system (cf. Exercise 2.3), which means that the left-hand sides are equal as tensors, and

$$\nabla_j \widetilde{V} = \mathsf{g}(\nabla_j \overline{V}, \overline{\cdot}). \tag{3.32}$$

In components (and in all coordinate systems),

$$V_i = g_{ik} V^k \tag{3.33}$$

$$V_{i;j} = g_{ik} V^k{}_{;j}. \tag{3.34}$$

The first equation here is just the component form of Eq. (3.29) (compare Eq. (2.14)). Note that the latter equation (which we obtained by comparing Eqs. (3.32) and (3.26)) is *not* trivial. From the properties of the metric we know that there exists *some* tensor that has components $A_{ij} = g_{ik} V^k{}_{;j}$: what this expression tells us is the nontrivial statement that this A_{ij} is exactly $V_{i;j}$.

That is to say that we did *not* get Eq. (3.34) by differentiating Eq. (3.33), though it looks rather as if we did. What *do* we get by differentiating Eq. (3.33)? By the Leibniz rule, Eq. (3.28),

$$V_{i;j} = g_{ik;j} V^k + g_{ik} V^k{}_{;j}.$$

But comparing this with Eq. (3.34), we see that the first term on the right-hand side must be zero, for arbitrary $\overline{V}$. Thus, in all coordinate systems (and relabelling),

$$g_{ij;k} = 0. \tag{3.35}$$

We have not exhausted the link between covariant differentiation and the metric. The two are related via

$$\Gamma^i_{jk} = \tfrac{1}{2} g^{il} (g_{jl,k} + g_{kl,j} - g_{jk,l}). \tag{3.36}$$

The proof is in Schutz §5.4, leading up to his equation (5.75); it is not long but involves, in Schutz's words, some 'advanced index gymnastics'. It depends on first proving that

$$\Gamma^k_{ij} = \Gamma^k_{ji}, \qquad \text{in all coordinate systems.} \tag{3.37}$$

Equation (3.36) completely cuts the link between the Christoffel symbols and cartesian coordinates, which might have lingered in your mind after Section 3.2.2 – once we have a metric, we can work out the Christoffel symbols' components immediately. [Exercises 3.12–3.14]

3.3 Covariant Differentiation in Curved Spaces

Having done all this work to develop covariant differentiation in flat space, but in purely geometrical terms, it might be a surprise to discover that there is

actually rather little to do to bring this over to the most general case of curved spaces. See Schutz §§6.2–6.4.

The first step is to define carefully the notion of a 'local inertial frame'.

3.3.1 Local Inertial Frames: The Local Flatness Theorem

Recall from Section 3.1.1 that a manifold is little more than a collection of points. What gives this manifold shape is the metric tensor g, which is a symmetric $\binom{0}{2}$ tensor which, in a particular coordinate system, has the components g_{ij}, which we can choose more or less how we like. In a different coordinate system, this same tensor will have different components $g_{\bar{\imath}\bar{\jmath}}$. The question is, can we find a coordinate system in which the metric has the particular form $\eta_{\bar{\imath}\bar{\jmath}} = \mathrm{diag}(-1, 1, 1, 1)$? That is, can we find a coordinate transformation $\Lambda^{\bar{\imath}}_{i}$ which transforms the coordinates x^i into the coordinates $x^{\bar{\imath}}$ in which the metric is diagonal?

If the *matrix* g_{ij} does not have three positive and one negative eigenvalues (i.e., a *signature* of $+3 - 1 = +2$), then no, we cannot, and the metric in question is uninteresting to us because it cannot describe our universe. If the metric does have a signature of $+2$, however, then it is a theorem of linear algebra that we can indeed find a transformation to coordinates in which the metric is diagonal *at a point*.

But we can do better than this. Recall that both g_{ij} and $\Lambda^{\bar{\imath}}_{i}$ are continuous functions of position; within the constraints that g be symmetric and Λ be invertible, they are arbitrary. By choosing the numbers $\Lambda^{\bar{\imath}}_{i}$ *and* their first derivatives, we can find coordinates that have their origin at P and in which

$$g_{\bar{\imath}\bar{\jmath}}\left(x^{\bar{k}}\right) = \eta_{\bar{\imath}\bar{\jmath}} + O\left(\left(x^{\bar{k}}\right)^2\right)$$

(compare Taylor's theorem), or

$$g_{\bar{\imath}\bar{\jmath}}(P) = \eta_{\bar{\imath}\bar{\jmath}} \tag{3.38a}$$

$$g_{\bar{\imath}\bar{\jmath},\bar{k}}(P) = 0 \tag{3.38b}$$

$$g_{\bar{\imath}\bar{\jmath},\bar{k}\bar{l}}(P) \neq 0. \tag{3.38c}$$

This is the *local flatness theorem*, and the coordinates $x^{\bar{k}}$ represent a *local inertial frame*, or LIF.

These coordinates are also known as 'normal' or 'geodesic' coordinates, and geodesics expressed in these coordinates have a particularly simple form. Also, in these coordinates, we can see from Eq. (3.36) that $\Gamma^{i}_{jk} = 0$ at P, which is just another way of saying that this space is locally flat.

Schutz proves the theorem at the end of his §6.2, and Carroll (2004) in §2.5; both are very illuminating.

3.3.2 The Covariant Derivative in Curved Spaces

You know how to differentiate things. For some function $f : \mathbb{R} \to \mathbb{R}$,

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}. \tag{3.39}$$

That's straightforward because it's obvious what $f(x+h)-f(x)$ means, and how we divide that by a number. Surely we can do a similar thing with vectors on a manifold. Not trivially, because remember that the vectors at P are not defined on the manifold but on the tangent plane $T_P(M)$ associated with that point, so the vectors at a different point Q are in a completely different space $T_Q(M)$, and so in turn it's not obvious how to 'subtract' one vector from the other. Differentiation on the manifold consists of finding ways to define just that 'subtraction'.

There are several ways to do this. One produces the 'Lie derivative', which is important in many respects, but which we will not examine.

The Lie derivative is a coordinate-independent derivative defined in terms of a vector field. A vector field $\overline{X}$ has *integral curves* such that, at each point p on the integral curve, the curve's tangent vector is $\overline{X}(p)$. As an example, stream lines in a fluid are integral curves of the fluid's velocity vector field. The Lie derivative of a function at a point p, written $(\pounds_{\overline{X}} f)_p$, is defined as the rate of change of the function along the (unique) integral curve of $\overline{X}$ going through p, and Lie derivatives of higher-order tensors are defined in an analogous way. The disadvantage of this type of derivative is that it clearly depends on an auxiliary vector field $\overline{X}$; but the compensating advantage is that it does *not* depend on a metric tensor, or any other definition of distance. These make it less useful than the covariant derivative for most GR applications, but it remains useful in other contexts, such as those where there is already an important vector field present, including applications in fluid dynamics. For details, see Stewart (1991) or Schutz (1980), or look at exercise 39 in Schutz's §6.9.

The other way to define this 'subtraction' uses the notion of 'parallel transport', which we define and examine now.

You *parallel transport* a vector along a curve by moving it so that the vectors at any two infinitesimally separated points are deemed parallel, in the broad sense of having the same length and pointing in the same direction. The precise rule for deciding whether two such vectors are parallel isn't specified here, and is broadly up to you, but we'll come back to that.

Figure 3.4 Parallel transporting a vector along a curve.

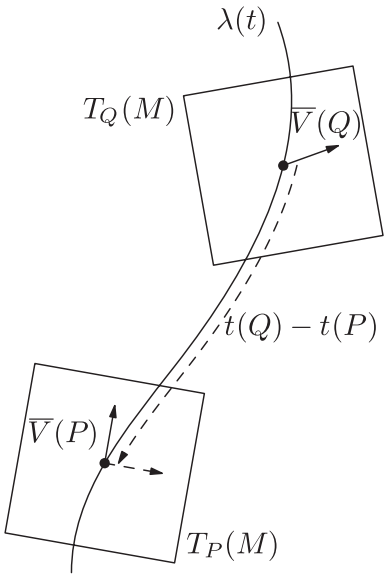

Figure 3.5 Pulling a vector from one tangent plane, attached to Q, to another, attached to P.

This gives us a way of talking about subtraction. Take a vector field $\overline{V}$ on the manifold, and two points P and Q that are both on some curve $\lambda(t)$, with tangent vector $\overline{U}$. We can take the vector $\overline{V}(Q)$ at Q and parallel transport it back to P; at that point it is in the same space $T_P(M)$ as the vector $\overline{V}(P)$ so we can unambiguously subtract them to give another vector in $T_P(M)$. These two points are a parameter distance $t(Q) - t(P)$ apart (which is a number), so we can divide the difference vector by that distance, find the limit as that distance goes to zero, and thus reconstruct all the components we need to define a differential just like Eq. (3.39). The differential we get by this process is the *covariant derivative of* $\overline{V}$ *along* $\overline{U}$, written $\nabla_{\overline{U}}\overline{V}$.

The covariant derivative depends on using parallel transport as a way of connecting vectors in two different tangent planes. The covariant derivative is sometimes also called the *connection* , and the Christoffel symbols the *connection coefficients* .

If $\overline{V}(Q)$ starts off as just the parallel-transported version of $\overline{V}(P)$, then when we parallel transport it back to P we'll get just $\overline{V}(P)$ again, so that this covariant derivative will be zero; thus

$$\nabla_{\overline{U}}\overline{V} = 0 \Leftrightarrow \quad (\overline{V} \text{ is parallel transported along } \overline{U}). \tag{3.40}$$

The crucial thing here is that nowhere in this account of the covariant derivative have we mentioned coordinates at all.

We've actually said rather little, here, because although this passage has, I hope, made clear how closely linked are the ideas of the covariant derivative and parallel transport, we haven't said how we go about choosing a definition of parallelism, and we haven't seen how this links to the covariant derivative we introduced in Section 3.2. The link is the locally flat LIF. Although the general idea of parallel transport, and in particular the definition I am about to introduce, may seem obvious or intuitive, do remember that there is an important element of arbitrariness in its actual definition.

Consider the coordinates representing the LIF at the point P. These are cartesian coordinates describing a flat space (but not euclidean, remember, since it does not have a euclidean metric). That means that the basis vectors are constant – their derivatives are zero. A definition of parallelism now jumps out at us: *two nearby vectors are parallel if their components in the LIF are the same*. But this is the definition of parallelism that was implicit in the differentiations we used in Sections 3.2.1 and 3.2.2, leading up to Eq. (3.18), and so the covariant derivative we end up with is the same one: the tensor $\nabla\overline{V}$ as defined in this section is the same as the covariant derivative of $\overline{V}$ in the LIF, by our choice of parallelism; and the covariant derivative in the (flat) LIF is the tensor $\nabla\overline{V}$ of Eq. (3.21).

Possibly rather surprisingly, we're now finished: we've already done all of the work required to define the covariant derivative in a curved space.

There are two further remarks remaining. Firstly, we can see that, in this cartesian frame, covariant differentiation is the same as ordinary differentiation, and so

$$V^i{}_{;j} = V^i{}_{,j} \qquad \text{in LIF.}$$

But this is true for any tensor, and so, specifically,

$$g_{ij;k} = g_{ij,k} = 0 \qquad \text{at } P,$$

by Eq. (3.38). But this is a tensor equation, so it is true in any coordinate system, and since there is nothing special about the point P, it is true at all points of the manifold:

$$g_{ij;k} = 0 \qquad \text{in any coordinate system.} \tag{3.41}$$

Secondly, as mentioned at the end of Section 3.2.3, from Eq. (3.41) we can deduce Eq. (3.36), since the conditions for that are still true in this more general case.

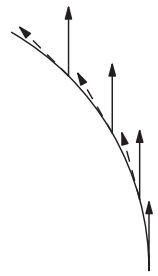

Figure 3.6 Not a geodesic: tangent vectors (dashed) and parallel-transported vectors.

The discussion in this section used less algebra than you may have expected for such a crucial part of the argument. Writing down the details of the construction of this derivative would be notationally intricate and take us a little too far afield. If you want details, there are more formal discussions of this mechanism via the notion of a 'pull-back map' in Schutz (1980, §6.3) or Carroll (2004, appendix A), and the covariant derivative is introduced in an axiomatic way in both Schutz (1980) and Stewart (1991). Also, the definition of parallelism via the LIF is not the only one possible, but picks out a particular derivative and set of connection coefficients, called the 'metric connection'. Only with this connection are Eq. (3.36) and Eq. (3.41) true. See also Schutz's discussion of geodesics on his pages 156–157, which elaborates the idea of parallelism introduced here.

[Exercise 3.15]

3.4 Geodesics

Consider a curve $\lambda(t)$ and its tangent vectors $\overline{U}$ (that is, the set of vectors $\overline{U}$ is a field that is defined at least at all the points along the curve λ). If we have another vector field $\overline{V}$, then the vector $\nabla_{\overline{U}}\overline{V}$ tells us how much $\overline{V}$ changes as we move along the curve λ to which $\overline{U}$ is the tangent. What happens if, instead of the arbitrary vector field $\overline{V}$, we take the covariant derivative of $\overline{U}$ itself? In general, $\nabla_{\overline{U}}\overline{U}$ will not be zero – if the curve 'turns a corner', then the tangent vector after the corner will no longer be parallel to the tangent before the corner. The meaning of 'parallel' here is exactly the same as the meaning of 'parallel' that was built in to the definition of the covariant derivative in the passage after Eq. (3.40). Curves that do not do this – that is, curves such that all the tangent vectors are parallel to each other – are the nearest thing to a straight line in the space, and indeed *are* a straight line in a flat space. A curve such as this is called a geodesic, more formally defined as follows:

$$\nabla_{\overline{U}}\overline{U} = 0. \Leftrightarrow \quad (\overline{U} \text{ is the tangent to a geodesic}) \tag{3.42}$$

Equation (3.42) has a certain spartan elegance, but if we are to do any calculations to discover what the path of the geodesic actually is, we need to unpack it.

The object $\nabla_{(\cdot)}\overline{U}$ is a $\binom{1}{1}$ tensor, as you will recall, with its vector argument denoted by the $(\cdot)$. Since it is a tensor, it is linear in this argument. That is, for any vector $\overline{A}$ and scalar a, $\nabla_{a\overline{A}}\overline{U} = a\nabla_{\overline{A}}\overline{U}$, and specifically $\nabla_{A^k\overline{e}_k}\overline{U} = A^k\nabla_{\overline{e}_k}\overline{U} \equiv A^k\nabla_k\overline{U}$. The vector $\overline{U}$ has components

$$\overline{U} = U^j\overline{e}_j,$$

and so Eq. (3.42) can be written

$$U^j\nabla_j\overline{U} = U^jU^i{}_{;j}\overline{e}_i = 0$$

(recalling Eq. (3.21)). The i-component of this equation is, using Eq. (3.18b),

$$U^jU^i{}_{;j} = U^jU^i{}_{,j} + U^jU^k\Gamma^i_{jk} = 0.$$

Let t be the parameter along the geodesic (that is, there is a parameterisation of the geodesic, $\lambda(t)$, with parameter t, which $\overline{U}$ is the tangent to). Then, using Eq. (3.5),

$$U^j = \overline{U}(\widetilde{\mathrm{d}}x^j) = \mathrm{d}x^j/\mathrm{d}t$$

and

$$U^i{}_{,j} = \partial/\partial x^j(\mathrm{d}x^i/\mathrm{d}t),$$

and we promptly find

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}x^i}{\mathrm{d}t}\right) + \Gamma^i_{jk}\frac{\mathrm{d}x^j}{\mathrm{d}t}\frac{\mathrm{d}x^k}{\mathrm{d}t} = 0. \tag{3.43}$$

This is the *geodesic equation*. For each i it is a second order differential equation with initial conditions comprising the initial position $x^i_0 = x^i(t_P)$ (if the parameter t has value t_P at point P) and initial direction/speed $U^i_0 = \mathrm{d}x^i/\mathrm{d}t|_{t_P}$. The theory of differential equations tells us that this equation does have a unique solution.

A parameter t for which we can write down the geodesic equation Eq. (3.43) is termed an *affine parameter*, and if t is an affine parameter, it is easy to confirm that $\phi = at + b$, where a and b are constants, is an affine parameter also.

An affine parameter is one that, in MTW's words (1973, §1.5), is 'defined so that motion looks simple'. You can reasonably measure time in seconds since midnight, or minutes (seconds/60), or minutes since noon (seconds/60 − 720). These are all affine transformations, and they share the property that unaccelerated motion is a linear function of time. If you were reckless enough to measure time in units of seconds-squared, then unaccelerated (that is, simple) motion would look very complicated indeed.

Another way of saying this is that *an affine parameter is the time coordinate of some inertial system* , and all that means it that an affine parameter is the time shown on some free-falling 'good' clock; it also means that a 'good' time is an affine transformation of another 'good' time. There are further remarks about affine parameters in Section 3.4.1.

The connection (or rather the class of connections) we have defined here (see Section 3.3.2) is constructed in such a way as to preserve parallelism. Such a connection is an *affine connection* – the word 'affine' comes from a Latin root meaning 'neighbouring'. Other types of connection are possible; see Schutz (1980, §6.14) for some challenging extra examples.

A geodesic is a curve of extremal length. In a space with a metric with the signature of GR, it is a curve of maximal length; in a euclidean space it is a curve of minimal length: for Euclid, a straight line is the shortest distance between two points.

Note on metric connections (extremely optional) : in other of these asides I have emphasised that this metric connection is not the only one definable. Since geodesics are defined in terms of the connection, it does indeed follow that the geodesics implied by these other connections are different from the geodesics of the metric connection, and specifically are not the curves of extremal length. This is bound up with the property Eq. (3.41), and the observation that only with the metric connection is the dot product $\mathsf{g}(\overline{A}, \overline{B})$ invariant under parallel transport. This is one reason why the metric connection is so important, to the point of being essentially ubiquitous in GR.

[Exercise 3.16]

3.4.1 The Variational Principle and the Geodesic Equation

We can prove directly that the geodesic is a curve of extremal length, by deriving the geodesic equation explicitly from a variational principle.

For a given curve through space-time, parameterised by λ, the length of the curve is given by

$$l = \int_{\text{curve}} \mathrm{d}s = \int_{\text{curve}} \left| g_{ij} \mathrm{d}x^i \mathrm{d}x^j \right|^{1/2} = \int_{\lambda_0}^{\lambda_1} \left| g_{ij} \dot{x}^i \dot{x}^j \right|^{1/2} \mathrm{d}\lambda, \equiv \int_{\lambda_0}^{\lambda_1} s \mathrm{d}\lambda,$$

where

$$s = \left| g_{ij} \dot{x}^i \dot{x}^j \right|^{1/2}$$

expresses the relationship between parameter distance and proper distance, and where dots indicate $\mathrm{d}/\mathrm{d}\lambda$. We wish to find a curve that is extremal, in the

sense that its length l is unchanged under first-order variations in the curve, for fixed λ_0 and λ_1. The *calculus of variations* (which as physicists you are most likely to have met in the context of classical mechanics) tells us that such an extremal curve $x^k(\lambda)$ is the solution of the Euler–Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial s}{\partial \dot{x}^k}\right) - \frac{\partial s}{\partial x^k} = 0.$$

Have a go, yourself, at deriving the geodesic equation from this before reading the rest of this section (at an appropriate point, you will need to restrict the argument to parameterisations of $s(\lambda)$ that are such that $\dot{s} = 0$.)

For s as given in the above equation, we find fairly directly that

$$-\frac{1}{2}\frac{\dot{s}}{s^2}2g_{kj}\dot{x}^j + \frac{1}{2s}\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(2g_{kj}\dot{x}^j\right) - \frac{1}{2s}g_{ij,k}\dot{x}^i\dot{x}^j = 0. \tag{3.44}$$

To simplify this, we can choose at this point to restrict ourselves to parameterisations of the curve that are such that $\mathrm{d}s/\mathrm{d}\lambda$ is constant along the curve, so that $\dot{s} = 0$; this λ is an affine parameter as described in the previous section. With this choice, and multiplying overall by s, we find

$$g_{kj,l}\dot{x}^j\dot{x}^l + g_{kj}\ddot{x}^j - \tfrac{1}{2}g_{ij,k}\dot{x}^i\dot{x}^j = 0$$

which, after relabelling and contracting with g^{kl}, and comparing with Eq. (3.36), reduces to

$$\ddot{x}^k + \Gamma^k_{ij}\dot{x}^i\dot{x}^j = 0, \tag{3.45}$$

the geodesic equation of Eq. (3.43).

As well as showing the direct connection between the geodesic equation and this deep variational principle, and thus making clear the idea that a geodesic is an extremal distance, this also confirms the significance of affine parameters that we touched on in Section 3.4. There is a 'geodesic equation' for non-affine parameters (namely Eq. (3.44)), but only when we choose an affine parameter λ does this equation take the relatively simple form of Eq. (3.43) or Eq. (3.45). The general solution of Eq. (3.44) is the same *path* as the geodesic, but because of the non-affine parameterisation it is not the same *curve* , and is not, formally, a geodesic. As we have dicussed before, the affine parameter of the geodesic is chosen so that motion looks simple.

Schutz discusses this at the very end of his §6.4, and the exercises corresponding to it.

3.5 Curvature

We now come, finally, to the coordinate-independent description of *curvature* . We approach it through the idea of parallel transport, as described in

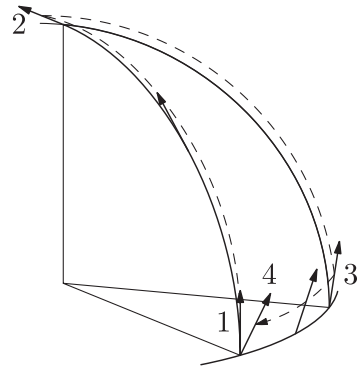

Figure 3.7 Dragging a vector across a sphere.

Section 3.3.2, and specifically through the idea of transporting a vector round a closed path. This section follows Schutz §6.5; MTW (1973) chapter 11 is illuminating on this.

First, it's important to have a clear picture of the way in which a vector will change as it is moved around a curved surface. In Figure 3.7 we see a view of a sphere and a vector that starts off on the equator, pointing 'north' (at 1). If we parallel transport this along the line of longitude until we get to the North Pole (2), then transport it south along a different line of longitude until we get back to the equator (3), then back along the equator to the point we started at (4), then we discover that the vector at the end does not end up parallel to the vector as it started. The vector on its round trip picks up information about the curvature of the surface; crucially this information is intrinsic to the surface, and does not depend on looking at the sphere from 'outside'. We now attempt to quantify this change, as a measure of the curvature of the surface.

3.5.1 The Riemann Tensor

Consider the path following lines of constant coordinate, in an arbitrary coordinate system. Figure 3.8 shows a loop in the plane of two coordinates x^σ and x^λ The line joining A and B, and the line from D to C, have coordinate x^σ varying along a line of constant x^λ, and lines B–C and A–D have x^λ varying along a line of constant x^σ. Now take a vector $\overline{V}$ at A, and parallel transport it to B, C, D, and back to A: how much has the vector changed during its circuit?

Parallel transporting the vector from A to B involves transporting $\overline{V}$ along the vector field $\overline{e}_\sigma$. From Eq. (3.40), this means that $\nabla_\sigma \overline{V} = 0$, or $V^i{}_{;\sigma} = 0$. That is (from Eq. (3.18b)),

$$\frac{\partial V^i}{\partial x^\sigma} = V^i{}_{,\sigma} = -\Gamma^i_{k\sigma} V^k. \tag{3.46}$$

Now, the components of the vector at B are

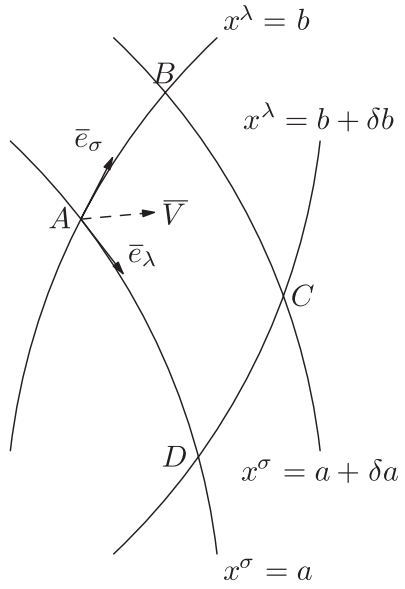

Figure 3.8 Taking a vector for a walk.

$$\begin{aligned} V^i(B) &= V^i(A) + \int_A^B \frac{\partial V^i}{\partial x^\sigma} \mathrm{d}x^\sigma \\ &= V^i(A) - \int_A^B \Gamma^i_{k\sigma} V^k \mathrm{d}x^\sigma \\ &= V^i(A) - \int_{x^\sigma = a}^{a+\delta a} \Gamma^i_{k\sigma} V^k \Big|_{x^\lambda = b} \mathrm{d}x^\sigma , \end{aligned}$$

where the integrand is evaluated along the line $\{x^\lambda = b\}$ from $x^\sigma = a$ to $x^\sigma = a + \delta a$. Doing the same thing for the other sides of the curve, we find:

$$\begin{aligned} \delta V^i = V^i(A_{\text{final}}) - V^i(A_{\text{init}}) = &- \int_{x^\sigma = a}^{a+\delta a} \Gamma^i_{j\sigma} V^j \Big|_{x^\lambda = b} \mathrm{d}x^\sigma \\ &- \int_{x^\lambda = b}^{b+\delta b} \Gamma^i_{j\lambda} V^j \Big|_{x^\sigma = a+\delta a} \mathrm{d}x^\lambda \\ &+ \int_{x^\sigma = a}^{a+\delta a} \Gamma^i_{j\sigma} V^j \Big|_{x^\lambda = b+\delta b} \mathrm{d}x^\sigma \\ &+ \int_{x^\lambda = b}^{b+\delta b} \Gamma^i_{j\lambda} V^j \Big|_{x^\sigma = a} \mathrm{d}x^\lambda . \end{aligned} \tag{3.47}$$

At this point we can take advantage of the fact that δa and δb are small by construction, ignore terms in δa^2 and δb^2, and thus take the integrands to be constant along the interval of integration (by expanding the integrand in a Taylor series, convince yourself that $\int_a^{a+\delta a} f(x)\mathrm{d}x = \delta a f(a) + O(\delta a^2)$).

We don't know what the $\Gamma^i_{j\lambda} V^j{}_{|x^\sigma = a+\delta a}$ and $\Gamma^i_{j\sigma} V^j{}_{|x^\lambda = b+\delta b}$ are (of course, since we are doing this calculation for perfectly general Γ), but since δa is small, we *can* estimate them using Taylor's theorem, finding

$$\Gamma^i_{j\lambda} V^j\big|_{x^\sigma = a+\delta a} = \Gamma^i_{j\lambda} V^j\big|_{x^\sigma = a} + \delta a \frac{\partial}{\partial x^\sigma} \Gamma^i_{j\lambda} V^j \bigg|_{x^\sigma = a} + O(\delta a^2)$$

(the $\partial/\partial x^\sigma$ is a derivative with respect to a single coordinate, which is why the σ index is correctly unmatched). Inserting this, and the similar expression involving δb, into Eq. (3.47), and ignoring terms of $O(\delta a^2, \delta b^2)$, we have

$$\begin{aligned} \delta V^i \approx & + \int_{x^\sigma = a}^{a+\delta a} \delta b \, \frac{\partial}{\partial x^\lambda} \Gamma^i_{j\sigma} V^j \bigg|_{x^\sigma = a, x^\lambda = b} \mathrm{d}x^\sigma \\ & - \int_{x^\lambda = b}^{b+\delta b} \delta a \, \frac{\partial}{\partial x^\sigma} \Gamma^i_{j\lambda} V^j \bigg|_{x^\sigma = a, x^\lambda = b} \mathrm{d}x^\lambda . \end{aligned}$$

However, the integrands here are now constant with respect to the variable of integration, so the integrals are easy:

$$\delta V^i \approx \delta a \delta b \left[\frac{\partial}{\partial x^\lambda} \left(\Gamma^i_{j\sigma} V^j \right) - \frac{\partial}{\partial x^\sigma} \left(\Gamma^i_{j\lambda} V^j \right) \right],$$

with all quantities evaluatated at the point A. If we now use Eq. (3.46) to get rid of the differentials of V^j, we find, to first order

$$\delta V^i = \delta x^\sigma \delta x^\lambda \left[\Gamma^i_{j\sigma,\lambda} - \Gamma^i_{j\lambda,\sigma} - \Gamma^i_{k\sigma} \Gamma^k_{j\lambda} + \Gamma^i_{k\lambda} \Gamma^k_{j\sigma} \right] V^j, \tag{3.48}$$

where we have written δa and δb as δx^σ and δx^λ respectively.

Let us examine this result. The left-hand side is the i component of a vector $\delta \overline{V}$ (we know this is a vector since it is the difference of two vectors located at the same point A; recall the vector-space axioms); we obtain that component δV^i by acting on the vector $\delta \overline{V}$ with the basis one-form $\widetilde{\omega}^i$. The right-hand side clearly depends on the vector $\overline{V}$ (also at the point A), whose components are V^j. The construction in Figure 3.8, which crucially has the area enclosed by constant-coordinate lines, depends on multiples of the basis vectors, $\delta a \, \overline{e}_\sigma$ and $\delta b \, \overline{e}_\lambda$. We can see that the *number* δV^i depends *linearly* on each of these four objects – one one-form and three vectors. This leads us to identify the numbers within the square brackets of Eq. (3.48) as the components of a $\binom{1}{3}$ tensor

$$R^i{}_{jkl} = \Gamma^i_{jl,k} - \Gamma^i_{jk,l} + \Gamma^i_{\sigma k} \Gamma^\sigma_{jl} - \Gamma^i_{\sigma l} \Gamma^\sigma_{jk} \tag{3.49}$$

(after some relabelling) called the *Riemann curvature tensor* (this notation, and in particular the overall sign, is consistent with Schutz; numerous other

conventions exist – see the discussion in Section 1.4.3). Thus Eq. (3.48) becomes

$$\delta V^i = R^i{}_{j\sigma\lambda} V^j \delta x^\sigma \delta x^\lambda . = \mathsf{R}(\widetilde{\omega}^i, \overline{V}, \delta x^\sigma \overline{e}_\sigma, \delta x^\lambda \overline{e}_\lambda). \tag{3.50}$$

This tensor tells us how the vector $\overline{V}$ varies after it is parallel transported on an arbitrary excursion in the area local to point A ('local', here, indicating small δa and δb); that is, it encodes all the information about the local shape of the manifold.

Another way to see the significance of the Riemann tensor is to consider the effect of taking the covariant derivative of a vector with respect to first one then another of the coordinates, $\nabla_i \nabla_j \overline{V}$. Defining the *commutator*

$$\left[\nabla_i, \nabla_j\right] V^k \equiv \nabla_i \nabla_j V^k - \nabla_j \nabla_i V^k, \tag{3.51}$$

it turns out that

$$\left[\nabla_i, \nabla_j\right] V^k = R^k{}_{lij} V^l \tag{3.52}$$

(see the 'dangerous-bend' paragraph in this section for details). This, or something like it, might not be a surprise. We discovered the Riemann tensor by taking a vector for a walk round the circuit $ABCDA$ in Figure 3.8 and working out how it changed as a result. The commutator Eq. (3.51) is effectively the result of taking a vector from A to C via B and via D, and asking how the two resulting vectors are different.

The Riemann tensor has a number of symmetries. In a locally inertial frame,

$$R^i{}_{jkl} = \tfrac{1}{2} g^{im} (g_{ml,jk} - g_{mk,jl} + g_{jk,ml} - g_{jl,mk}), \tag{3.53}$$

and so

$$R_{ijkl} \equiv g_{im} R^m{}_{jkl} = \tfrac{1}{2} (g_{il,jk} - g_{ik,jl} + g_{jk,il} - g_{jl,ik}). \tag{3.54}$$

Note that this is *not* a tensor equation, even in these coordinates: in such inertial coordinates $V^i{}_{,j} = V^i{}_{;j}$ and so an expression involving single partial derivatives of inertial coordinates can be trivially rewritten as a (covariant) tensor equation by simply rewriting the commas as semicolons; however the same is *not* true of second derivatives, so that Eq. (3.54) does not trivially correspond to a covariant expression.[3]

[3] To see this, consider differentiating ∇ twice, and note that $V^k{}_{;ln}$ includes a term in $\Gamma^k_{ml,n}$. This term is non-zero (in general) even in locally flat coordinates, which means that $V^k{}_{;ln}$ does not reduce to $V^k{}_{,ln}$ in the LIF. That means, in turn, that Eq. (3.53) cannot be taken to be the LIF version of a covariant equation, and thus that we cannot obtain a covariant expression for the Riemann tensor by swapping these commas with semicolons. Compare also Eq. (3.38c), and the argument leading up to Eq. (3.36).

By simply permuting indexes in Eq. (3.54), you can see that

$$R_{ijkl} = -R_{jikl} = -R_{ijlk} = R_{klij} \tag{3.55a}$$

$$R_{ijkl} + R_{iljk} + R_{iklj} = 0. \tag{3.55b}$$

These , finally are tensor equations so that (as usual) although we worked them out in a particular coordinate system, they are true in all coordinate systems, and tell us about the symmetry properties of the underlying geometrical object.

Notice that the definition of the Riemann tensor in Eq. (3.52) does *not* involve the metric; this is not a coincidence. The development of the Riemann tensor up to Eq. (3.50) is (I think) usefully explicit, but also rather laborious, and manifestly involves the Christoffel symbols, which you are possibly used to thinking of as being very closely related to the metric. That's not false, but another way of getting to the Riemann tensor is to define the covariant derivative $\nabla_{\overline{U}}\overline{V}$ as an almost immediate consequence of the definition of parallel transport, and then notice that the operator

$$[\nabla_{\overline{U}}, \nabla_{\overline{V}}] - \nabla_{[\overline{U},\overline{V}]} = \mathsf{R}(\overline{U}, \overline{V})$$

maps vectors to vectors, defining the $\binom{1}{3}$ Riemann tensor R (this is *not* a derivation; see Schutz (1980, §6.8)). The point here is that R is not just a function of the metric, but is a completely separate tensor, even though, in the case of a metric connection, it is calculable in terms of the metric via expressions such as Eq. (3.53). This is why it is not a contradiction (though I agree it is surprising at first thought) that the Riemann tensor starts off with considerably more degrees of freedom than the metric tensor. There are some further remarks about the number of degrees of freedom in the Riemann tensor in Section 4.2.5.

[Exercises 3.17–3.19]

3.5.2 Geodesic Deviation

In Section 1.2.4, we briefly imagined two objects in free fall near the earth (Figure 1.5), and noted that the distance between them would decrease as they both moved towards the centre of the earth. We are now able to state that these free-falling objects are following geodesics in the spacetime surrounding the earth, which is curved as a result of the earth's mass (though we cannot say much more than this without doing the calculation, which is a bit of physics which we learn about in the next chapter). We see, then, that the effect of the curvature of space-time is to cause the distance between these two geodesics to decrease; this is known as *geodesic deviation* , and we are now in a position to see how it relates to curvature.

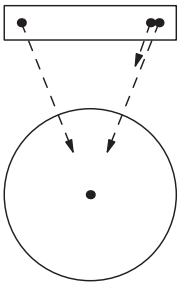

Figure 3.9 Geodesics on earth.

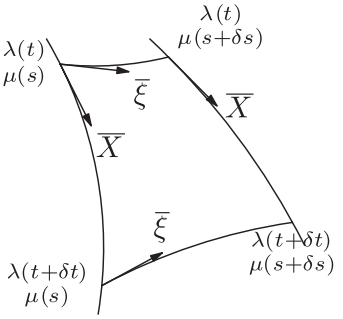

Figure 3.10 Geodesic deviation.

Schutz covers this at the end of his section 6.5, using a different argument from the following. I plan to describe it in a different style here, partly because a more geometrically minded explanation makes a possibly welcome change from relentless components.

First, some useful formulae. (i) Marginally rewriting Eq. (3.52), we find

$$\left[\nabla_{\overline{X}}, \nabla_{\overline{Y}}\right]\overline{V} = X^j Y^l \left[\nabla_j, \nabla_l\right] V^k \overline{e}_k = R^k{}_{ijl} V^i X^j Y^l \overline{e}_k. \tag{3.56}$$

(ii) Using the commutator $\left[\overline{A}, \overline{B}\right] \equiv \overline{A}\,\overline{B} - \overline{B}\,\overline{A}$, we find

$$\nabla_{\overline{A}}\overline{B} - \nabla_{\overline{B}}\overline{A} = \left[\overline{A}, \overline{B}\right], \tag{3.57}$$

which is proved in Exercise 3.20.

Consider two sets of curves, $\lambda(t)$ corresponding to a field of tangent vectors $\overline{X}$, and $\mu(s)$ with tangent vectors $\overline{\xi}$, and suppose that, in some region of the manifold, they cross each other (see Figure 3.10). Choose the curves and their parameterisation such that each of the λ curves is a curve of constant s and each of the μ curves is a curve of constant t. Thus, specifically, the $\overline{\xi}$ vector – known as the *connecting vector* – joins points on two λ curves that have the same parameter t. What we have actually described, here, is (part of) a set of coordinate functions; you will see that the curves λ and μ have exactly the properties that the conventionally written coordinate functions x^i

have. Because of this construction, it does not matter in which order we take the derivatives $\mathrm{d}/\mathrm{d}t$ and $\mathrm{d}/\mathrm{d}s$, so that

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}s} = \frac{\mathrm{d}}{\mathrm{d}s}\frac{\mathrm{d}}{\mathrm{d}t} \Leftrightarrow \left[\frac{\mathrm{d}}{\mathrm{d}t}, \frac{\mathrm{d}}{\mathrm{d}s}\right] = 0,$$

or, since $\overline{X} = \mathrm{d}/\mathrm{d}t$ and $\overline{\xi} = \mathrm{d}/\mathrm{d}s$,

$$\left[\overline{X}, \overline{\xi}\right] = 0.$$

Thus, referring to Eq. (3.57),

$$\nabla_{\overline{X}}\overline{\xi} = \nabla_{\overline{\xi}}\overline{X}. \tag{3.58}$$

Now suppose particularly that the curves $\lambda(t)$ are geodesics, which means that $\nabla_{\overline{X}}\overline{X} = 0$. Then the vector $\overline{\xi}$ joins points on the two geodesics that have the same affine parameter.

That means that the second derivative of $\overline{\xi}$ carries information about how quickly the two geodesics are accelerating apart (note that this is 'acceleration' in the sense of 'second derivative of position coordinate', and not anything that would be measured by an accelerometer – observers on the two geodesics would of course experience *zero* physical acceleration). Using Eq. (3.58) the calculation is easy. The second derivative is

$$\nabla_{\overline{X}}\nabla_{\overline{X}}\overline{\xi} = \nabla_{\overline{X}}\nabla_{\overline{\xi}}\overline{X} = \nabla_{\overline{\xi}}\nabla_{\overline{X}}\overline{X} + R^k{}_{ijl}X^iX^j\xi^l\overline{e}_k, \tag{3.59}$$

where the first equality comes from Eq. (3.58) and the second from Eq. (3.56). The first term on the right-hand side disappears since $\nabla_{\overline{X}}\overline{X} = 0$ along a geodesic. Now, the covariant derivative with respect to the vector $\overline{X}$ is just the derivative with respect to the geodesic's parameter t (since λ is part of a coordinate system; see Section 3.2.2), so that this equation turns into

$$\left(\frac{\mathrm{d}^2\overline{\xi}}{\mathrm{d}t^2}\right)^k = R^k{}_{ijl}X^iX^j\xi^l. \tag{3.60}$$

Thus the amount by which two geodesics diverge depends on the curvature of the space they are passing through. Note that the left-hand side here is the k-component of the second derivative of the vector $\overline{\xi}$, and is a conventional shortcut for $\nabla_{\overline{X}}\nabla_{\overline{X}}$; it is *not* the second derivative of the ξ^k component $\mathrm{d}^2\xi^k/\mathrm{d}t^2$, though some books (e.g., Stewart (1991, §1.9)) rather confusingly write it this way. [Exercises 3.20–3.22]

Exercises

Most of the exercises in Schutz's §6.8 should be accessible.

Exercise 3.1 (§ 3.1.3) By considering the contraction of the gradient with a vector $a\mathrm{d}/\mathrm{d}t + b\mathrm{d}/\mathrm{d}s$, show that the gradient one-form defined by Eq. (3.5) is a linear function of its argument, and therefore a valid one-form.

Exercise 3.2 (§ 3.1.3) Given a potential field ϕ in a 2-d space, show immediately that the gradient of the field, in polar coordinates (r, θ), is

$$\widetilde{\mathrm{d}}\phi = \frac{\partial\phi}{\partial r}\widetilde{\mathrm{d}}r + \frac{\partial\phi}{\partial\theta}\widetilde{\mathrm{d}}\theta$$

(once you've done that, expand $\widetilde{\mathrm{d}}\phi(\overline{V})$, where $\overline{V} = V^r\overline{e}_r + V^\theta\overline{e}_\theta$, and persuade yourself that you expected the result).

The metric in polar coordinates is $g_{ij} = \mathrm{diag}(1, r^2)$, and correspondingly $g^{ij} = \mathrm{diag}(1, 1/r^2)$. Thus calculate the *vector* gradient of the field ϕ, namely the vector $\overline{\mathrm{d}\phi}$.

This may not look quite as you expected, but recall that $\overline{e}_r$ and $\overline{e}_\theta$ here form a coordinate basis, and so are slightly different from the 'usual' basis of polar coordinates $\hat{r} = \overline{e}_r$ and $\hat{\theta} = (1/r)\overline{e}_\theta$. Immediately calculate $\overline{\mathrm{d}\phi}$ in this basis, and confirm that this looks more as you might expect.

Exercise 3.3 (§ 3.1.4) In the $\{x, y\}$ cartesian coordinate system (with basis vectors $\partial/\partial x$ and $\partial/\partial y$), the metric is simply $\mathrm{diag}(1, 1)$. Consider a new coordinate system $\{u, v\}$, (with basis $\partial/\partial u$ and $\partial/\partial v$), defined by

$$\begin{aligned} u &= \tfrac{1}{2}(x^2 - y^2) \\ v &= xy. \end{aligned} \tag{i}$$

You might also want to look back at the 'dangerous bend' paragraphs below Eq. (2.23).

(a) Write $x^1 = x$, $x^2 = y$, $x^{\bar{1}} = u$, $x^{\bar{2}} = v$, and thus, referring to Eq. (3.9), calculate the matrices $\Lambda^{\bar{i}}_{j}$ and $\Lambda^{i}_{\bar{j}}$. (The easiest way of doing the latter calculation is to calculate $\partial u/\partial u$, $\partial u/\partial v$, ..., and solve for $\partial x/\partial u$, $\partial x/\partial v$, ..., ending up with expressions in terms of x, y, and $r^2 = x^2 + y^2$.)

(b) From Eq. (2.24),

$$g_{\bar{i}\bar{j}} = \Lambda^{i}_{\bar{i}}\Lambda^{j}_{\bar{j}}g_{ij}.$$

Thus calculate the components $g_{\bar{i}\bar{j}}$ of the metric in terms of the coordinates $\{u, v\}$ (you can end up with expressions in terms of u and v, via $4(u^2 + v^2) = r^4$).

(c) A one-form has cartesian coordinates (A_x, A_y) and coordinates (A_u, A_v) in the new coordinate system. Show that

$$A_u = \frac{xA_x - yA_y}{x^2 + y^2},$$

and derive the corresponding expression for A_v. $[d^+u^+]$

Exercise 3.4 (§ 3.1.4) (a) Write down the expressions for cartesian coordinates $\{x, y\}$ as functions of polar coordinates $\{r, \theta\}$, thus calculate $\partial x/\partial r$, $\partial x/\partial\theta$, $\partial y/\partial r$ and $\partial y/\partial\theta$, and thus find the components of the transformation matrix from cartesian to polar coordinates, Eq. (3.9b).

(b) The inverse transformation is

$$r^2 = x^2 + y^2, \qquad \theta = \arctan\left(\frac{y}{x}\right).$$

Differentiate these, and thus obtain the inverse transformation matrix Eq. (3.9a). Verify that the product of these two matrices is indeed the identity matrix. Compare Section 2.3.2.

(c) Let $\overline{V}$ be a vector with cartesian coordinates $\{x, y\}$, so that

$$\overline{V} = x\overline{e}_x + y\overline{e}_y.$$

Show that $\dot{\overline{V}}$ and $\ddot{\overline{V}}$ have components $\{\dot{x}, \dot{y}\}$ and $\{\ddot{x}, \ddot{y}\}$ in this basis.

(d) Using the relations $x = r\cos\theta$ and $y = r\sin\theta$, write down expressions for x, y, $\dot{x}$ and $\dot{y}$ in terms of polar coordinates r and θ and their time derivatives.

(e) Now use the general transformation law Eq. (3.9a)

$$V^{\bar{\imath}} = \Lambda^{\bar{\imath}}_j V^j = \frac{\partial x^{\bar{\imath}}}{\partial x^j} V^j$$

to transform the components of the vectors $\dot{\overline{V}}$ and $\ddot{\overline{V}}$, which you obtained in (c), into the polar basis $\{\overline{e}_r, \overline{e}_\theta\}$, and show that

$$\dot{\overline{V}} = r\overline{e}_r + \theta\overline{e}_\theta$$
$$\ddot{\overline{V}} = \left(\ddot{r} - r\dot{\theta}^2\right)\overline{e}_r + \left(\ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta}\right)\overline{e}_\theta.$$

$[u^{++}]$

Exercise 3.5 (§ 3.1.4) Define a scalar field, ϕ, by

$$\phi(x, y) = x^2 + y^2 + 2xy,$$

for cartesian coordinates $\{x, y\}$.

(a) From Eq. (3.5), the ith component of the gradient one-form $\widetilde{\mathrm{d}\phi}$ is obtained by taking the contraction of the gradient with the basis vector $\overline{e}_{\bar{i}} = \partial/\partial x^{\bar{i}}$. Thus write down the components of the gradient one-form with respect to the cartesian basis.

(b) The result of Exercise 2.10 says that the transformation law for the components of a one-form is

$$A_{\bar{i}} = \Lambda^{j}_{\bar{i}} A_j = \frac{\partial x^j}{\partial x^{\bar{i}}} A_j .$$

Thus determine the components of $\widetilde{\mathrm{d}\phi}$ in polar coordinates $\{r, \theta\}$.

(c) By expressing ϕ in terms of r and θ, obtain directly the polar components of $\widetilde{\mathrm{d}\phi}$ and verify that they agree with those obtained in (b).

(d) Write down the components of the metric tensor in cartesian coordinates, g_{xx}, g_{xy}, g_{yx}, g_{yy}, and by examining Eq. (2.15), write down the components of the metric tensor with raised indexes, g^{xx}, g^{xy}, g^{yx}, g^{yy}. Hence determine the cartesian components of the *vector* gradient $\overline{\mathrm{d}\phi}$ (i.e., with raised index).

(e) Recall the metric for polar coordinates, and thus the components g^{rr}, $g^{r\theta}$, $g^{\theta r}$, and $g^{\theta\theta}$. Hence determine the polar components of $\overline{\mathrm{d}\phi}$. Comment on the answers to parts (d) and (e). [$d^+ u^+$]

Exercise 3.6 (§ 3.2.1) In Eq. (3.16) we see, for example, two lowered θs on the left-hand side with no θ on the right-hand side. Why isn't this this an Einstein summation convention error?

Exercise 3.7 (§ 3.2.2) Consider a vector field $\overline{V}$ with cartesian components $\{V^x, V^y\} = \{x^2 + 3y, y^2 + 3x\}$.

(a) Using the transformation law for a $\binom{1}{0}$ tensor, and the result of Exercise 3.4, determine $\{V^r, V^\theta\}$, the components of the same vector field $\overline{V}$ with respect to the polar basis $\{\overline{e}_r, \overline{e}_\theta\}$.

(b) Write down the components of the covariant derivative $V^i{}_{;j}$ in cartesian coordinates.

(c) Using the fact that $V^i{}_{;j}$ transforms as a $\binom{1}{1}$ tensor, compute the components of the covariant derivative with respect to the polar coordinate basis by transforming the $V^i{}_{;j}$ obtained in part (b).

(d) Now, taking a different tack, compute the polar components of the covariant derivative of $\overline{V}$, by differentiating the polar coordinates obtained in (a). That is, use Eq. (3.18b) and the Christoffel symbols for polar coordinates, Eq. (3.15).

(e) Verify that the polar components obtained in (c) and (d) are the same.

[u^+]

Exercise 3.8 (§ 3.2.2) Do Exercise 3.7 again, but this time working with the one-form field $\widetilde{A}$, with cartesian components $\{x^2 + 3y, y^2 + 3x\}$.

Exercise 3.9 (§ 3.2.2) Comparing Exercises 3.7 and 3.8, verify that in both cartesian and polar coordinates $g_{ik}V^k{}_{;j} = A_{i;j}$. [$d^-$]

Exercise 3.10 (§ 3.2.2) Suppose that $\overline{U}$ is tangent to a geodesic, thus $\nabla_{\overline{U}}\overline{U} = 0$. Consider another vector field $\overline{V}$, related to $\overline{U}$ by $\overline{V} = a\overline{U}$, where a is a scalar field. By recalling that $(\nabla \overline{V})^\alpha{}_\beta = V^\alpha{}_{;\beta} = V^\alpha{}_{,\beta} + \Gamma^\alpha_{\beta\gamma}V^\gamma$, show that $\overline{V}$ is also tangent to a geodesic, if and only if a is a constant. [u^+]

Exercise 3.11 (§ 3.2.2) Deduce Eq. (3.26). Use the Leibniz rule $\nabla_{\overline{V}}\langle\widetilde{p}, \overline{A}\rangle = \langle\nabla_{\overline{V}}\widetilde{p}, \overline{A}\rangle + \langle\widetilde{p}, \nabla_{\overline{V}}\overline{A}\rangle$, and Eq. (3.24) acting on $f = \langle\widetilde{p}, \overline{A}\rangle$. [$u^+$]

Exercise 3.12 (§ 3.2.3) Derive Eq. (3.33) from Eq. (3.29) (one-liner). Derive Eq. (3.34) from Eq. (3.32) (few lines). [d^-]

Exercise 3.13 (§ 3.2.3) Let A_j be the components of an arbitrary one-form. Write down the transformation law for A_j and for its covariant derivative $A_{j;k}$. We can obtain an expression for $A_{\bar{j};\bar{k}}$ either by differentiating $A_{\bar{j}}$, or by transforming $A_{j;k}$. By comparing the two expressions for $A_{\bar{j};\bar{k}}$, show that the transformation law for the Christoffel symbols has the form

$$\Gamma^{\bar{i}}_{\bar{j}\bar{k}} = \frac{\partial x^{\bar{i}}}{\partial x^i}\frac{\partial x^j}{\partial x^{\bar{j}}}\frac{\partial x^k}{\partial x^{\bar{k}}}\Gamma^i_{jk} + \frac{\partial x^{\bar{i}}}{\partial x^l}\frac{\partial^2 x^l}{\partial x^{\bar{j}}\partial x^{\bar{k}}}.$$

The first term in this looks like Eq. (2.24); but the second term will not be zero in general, demonstrating that Christoffel symbols are not the components of any tensor. [d^+]

Exercise 3.14 (§ 3.2.3) Suppose that in one coordinate system the Christoffel symbols are symmetric in their lower indexes, $\Gamma^i_{jk} = \Gamma^i_{kj}$. By considering the transformation law for the Christoffel symbols, obtained in Exercise 3.13, show that they will be symmetric in any coordinate system.

Exercise 3.15 (§ 3.3.2) Things to think about: Why have you never had to learn about covariant differentiation before now? The glib answer is, of course, that you weren't learning GR; but what was it about the vector calculus that you did learn that meant you never had to know about connection coefficients? Or, given that you did effectively learn about them, but didn't know that was what they were called, why do we have to go into so much more detail about them now? There are a variety of answers to these questions, at different levels.

Exercise 3.16 (§ 3.4) (a) On the surface of a sphere, we can pick coordinates θ and ϕ, where θ is the colatitude, and ϕ is the azimuthal coordinate. The components of the metric in these coordinates are

$$g_{\theta\theta} = 1, \qquad g_{\phi\phi} = \sin^2\theta, \qquad \text{others zero.}$$

Show that the components of the metric with raised indexes are

$$g^{\theta\theta} = 1, \qquad g^{\phi\phi} = \frac{1}{\sin^2\theta}, \qquad \text{others zero.}$$

(b) The Christoffel symbols are defined as

$$\Gamma^i_{kl} = \tfrac{1}{2} g^{ij} (g_{jk,l} + g_{jl,k} - g_{kl,j}),$$

and the geodesic equation is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}x^i}{\mathrm{d}t}\right) + \Gamma^i_{jk}\frac{\mathrm{d}x^j}{\mathrm{d}t}\frac{\mathrm{d}x^k}{\mathrm{d}t} = 0,$$

for a geodesic with parameter t. Using these find the Christoffel symbols for these coordinates (i.e., $\Gamma^\theta_{\theta\theta}$, $\Gamma^\theta_{\theta\phi}$ and so on), and thus show that the geodesic equations for these coordinates are

$$\ddot{\theta} - \sin\theta\cos\theta\,\dot{\phi}^2 = 0 \tag{i}$$

$$\ddot{\phi} + 2\frac{\cos\theta}{\sin\theta}\dot{\theta}\dot{\phi} = 0, \tag{ii}$$

where dots indicate differentiation with respect to the parameter t.

(c) Using the result of part (b), or any other properties of geodesics that you know, explain, giving reasons, which of the following curves are geodesics, for affine parameter t.

1. $\phi = t, \theta = \pi/2$ 2. $\phi = t, \theta = \pi/4$ 3. $\phi = t, \theta = 0$
4. $\phi = t, \theta = t$ 5. $\phi = \phi_0, \theta = t$ 6. $\phi = \phi_0, \theta = 2t - 1$
7. $\phi = \phi_0, \theta = t^2$

(d) The surface of the sphere can also be described using the coordinates (x, y) of the *Mercator projection* , as used on some maps of the world to represent the surface in the form of a rectangular grid. The coordinates (x, y) can be given as a function of the coordinates (θ, ϕ) listed in this problem.

If you were to perform the same calculation as above using these coordinates, would you obtain the same Christoffel symbols? Explain your answer. Comment on the relationship between the curves in part (c) and the geodesics obtained using the Mercator coordinates. [d^+u^{++}]

Exercise 3.17 (§ 3.5.1) Prove Eq. (3.52). Write $\nabla_i \nabla_j V^k = \nabla_i (V^k{}_{;j}) = (V^k{}_{;j})_{;i}$, and use the expression Eq. (3.27) to expand the derivative with respect to x^i. At this point, decide to work in LIF coordinates, in which all the $\Gamma^k_{ij} = 0$, making the algebra easier. Thus deduce that $\nabla_i \nabla_j V^k = V^k{}_{,ji} + \Gamma^k_{lj,i} V^l$. You can then immediately write down an expression for $\nabla_j \nabla_i V^k$. Subtract these two expressions (to form $[\nabla_i, \nabla_j] V^k$), noting that the usual partial differentiation of components commutes: $V^k{}_{,ij} = V^k{}_{,ji}$. Compare the result with the definition of the Riemann tensor in Eq. (3.49), and arrive at Eq. (3.52). $[d^+u^{++}]$

Exercise 3.18 (§ 3.5.1) Prove Eq. (3.53). Expand the definition of the Riemann tensor in Eq. (3.49) in the local inertial frame, in which $g_{kl,m} = 0$ (Eq. (3.38)). Recall that partial derivatives always commute.

Exercise 3.19 (§ 3.5.1) In Exercise 3.16, you calculated the Christoffel symbols for the surface of the unit sphere. Calculate the components of the curvature tensor for these coordinates, plus the Ricci tensor $R_{j\gamma} = R^i{}_{ji\gamma}$ and the Ricci scalar $R = g^{j\gamma} R_{j\gamma}$ (see Chapter 4).

You can most conveniently do this by calculating selected components of the curvature tensor R_{ijkl} obtained by lowering the first index on Eq. (3.49); you can cut down the number of calculations you need to do by using the symmetry relations Eq. (3.55) heavily. Why should you not use Eq. (3.54), which appears to be more straightforward?

This question is long-winded rather than terribly hard. It's worthwhile slogging through it, however, since it gives valuable practice handling indices, and makes the idea of the curvature tensor rather more tangible. $[d^-u^+]$

Exercise 3.20 (§ 3.5.2) Prove Eq. (3.57), by writing it in component form. Recall Eq. (3.37). The last step is the tricky bit, but recall that for a (tangent) vector $\overline{A}$, $\overline{A}f = A^k \overline{e}_k f = A^k f_{,k}$, where f is any function, including a vector component.

Exercise 3.21 (§ 3.5.2) Consider coordinates on a sphere, as you did in Exercise 3.16, and consider the geodesics $\lambda(t)$ in Figure 3.11 with affine parameter t and tangent vectors $\overline{X}$ – these are great circles through the poles. The curves $\mu(s)$ with tangent vectors $\overline{\xi}$ are connecting curves as discussed in Section 3.5.2.

We can parameterise the curve $\lambda(t)$ using the coordinates (θ, ϕ), as

$$\lambda(t) : \theta\big(\lambda(t)\big) = t, \qquad \phi\big(\lambda(t)\big) = \phi_0$$

(compare Section 3.1.2), and you verified in Exercise 3.16 that this does indeed satisfy the geodesic equation.

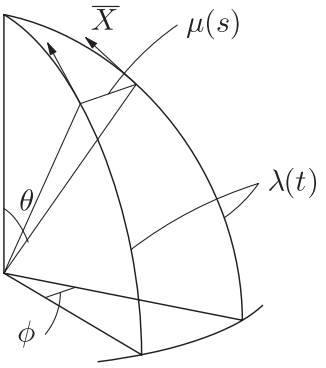

Figure 3.11 Geodesics on a sphere.

(a) Using Eq. (3.1), show that the components of $\overline{X}$ are

$$X^\theta = 1, \qquad X^\phi = 0.$$

(b) Write Eq. (3.60) as

$$g_{\lambda k}(\nabla_{\overline{X}}\nabla_{\overline{X}}\overline{\xi})^k - g_{\lambda k}R^k{}_{ijl}X^iX^j\xi^l = 0 \tag{i}$$

and, by using the components of the curvature tensor that you worked out in Exercise 3.19, show that

$$(\nabla_{\overline{X}}\nabla_{\overline{X}}\overline{\xi})^\theta = 0 \tag{iia}$$

$$(\nabla_{\overline{X}}\nabla_{\overline{X}}\overline{\xi})^\phi + \xi^\phi = 0. \tag{iib}$$

This tells us that the connecting vector – the tangent vector to the family of curves $\mu(s)$, connecting points of equal affine parameter along the geodesics $\lambda(t)$ – does not change its θ component, but does change its ϕ component. Which isn't much of a surprise.

(c) Can we get more out of this? Yes, but to do that we have to calculate $\nabla_{\overline{X}}\nabla_{\overline{X}}\overline{\xi}$, which isn't quite as challenging as it might look. From Eq. (3.21) we write

$$\nabla_{\overline{X}}\overline{\xi} = X^i\nabla_i\overline{\xi} = X^i\overline{e}_j\xi^j{}_{;i} = X^i\overline{e}_j\left(\frac{\partial\xi^j}{\partial x^i} + \Gamma^j_{i\gamma}\xi^\gamma\right). \tag{iii}$$

You have worked out the Christoffel symbols for these coordinates in Exercise 3.16, so we *could* trundle on through this calculation, and find expressions for the components of the connecting vector $\overline{\xi}$ from Eq. (ii). In order to illustrate something useful in a reasonable amount of time, however, we will short-circuit that by using our previous knowledge of this coordinate system.

The curve

$$\mu(s) : \theta(s) = \theta_0, \quad \phi(s) = s$$

is *not* a geodesic (it is a small circle at colatitude θ_0), but it does connect points on the geodesics $\lambda(t)$ with equal affine parameter t; it is a connecting curve for this family of geodesics. Convince yourself of this and, as in part (a), satisfy yourself that the tangent vector to this curve, $\xi = \mathrm{d}/\mathrm{d}s$, has components $\xi^\theta = 0$ and $\xi^\phi = 1$; and use this together with the components of the tangent vector $\overline{X}$ and the expression Eq. (iii) to deduce that

$$\dot{\overline{\xi}} \equiv \nabla_{\overline{X}}\overline{\xi} = 0\overline{e}_\theta + \cot\theta\overline{e}_\phi,$$

(where $\dot{\overline{\xi}}$ is simply a convenient – and conventional – notation for $\nabla_{\overline{X}}\overline{\xi}$) or $\xi^\theta = 0$, $\xi^\phi = \cot\theta$.

(d) So far so good. In exactly the same way, take the covariant derivative of $\dot{\overline{\xi}}$, and discover that

$$\nabla_{\overline{X}}\dot{\overline{\xi}} = \nabla_{\overline{X}}\nabla_{\overline{X}}\overline{\xi} = 0\overline{e}_\theta - 1\overline{e}_\phi = -\overline{\xi},$$

and note that this $\overline{\xi}$ does in fact accord with the geodesic deviation equation of Eq. (ii).

Note that this example is somewhat fake, in that, in (c), we set up the curve $\mu(s)$ as a connecting curve, and all we have done here is verify that this was consistent. If we were doing this for real, we would not know (all of) the components of $\overline{\xi}$ beforehand, but would carry on differentiating $\overline{\xi}$ as we started to do in (c), put the result into the differential equation Eq. (ii) and thus deduce expressions for the components ξ^k.

As a final point, note that the *length* of the connecting vector $\overline{\xi}$ is just

$$\mathsf{g}(\overline{\xi},\overline{\xi}) = g_{ij}\xi^i\xi^j = \sin^2\theta,$$

which you could possibly have worked out from school trigonometry (but it wouldn't have been half so much fun). $[u^+]$

Exercise 3.22 (§ 3.5.2) In the newtonian limit, the metric can be written as

$$g_{ij} = \eta_{ij} + h_{ij} \tag{i}$$

where

$$\eta_{ij} = \mathrm{diag}(-1, 1, 1, 1)$$

$$h_{ij} = \begin{cases} -2\phi & i = j \\ 0 & i \neq j \end{cases},$$

and ϕ is the newtonian gravitational potential $\phi(r) = GM/r$. In this limit, and with this metric, the curvature tensor can be written as

$$2R_{ijkl} = h_{il,jk} + h_{jk,il} - h_{ik,jl} - h_{jl,ik}.$$

The equation for geodesic deviation is

$$\frac{\mathrm{d}^2\xi^i}{\mathrm{d}t^2} = R^i{}_{jkl}U^jU^k\xi^l, \tag{ii}$$

where the vectors $\overline{U}$ are tangent to geodesics, and we can take them to be velocity vectors.

Consider two particles in free fall just above the Earth's North Pole, so that their (cartesian) coordinates are both approximately $x = y = 0$, $z = R$, where R is the radius of the Earth. Take them to be separated by a separation vector $\overline{\xi} = (0, \xi^x, 0, 0)$, where $\xi^x \ll R$. Since they are falling along geodesics, their velocity vectors are both approximately $\overline{U} = (U^t, 0, 0, U^z)$.

With this information, show that the two particles accelerate towards each other such that

$$\frac{\mathrm{d}^2\xi^x}{\mathrm{d}t^2} = -\frac{GM}{r^3}\xi^x \tag{iii}$$

to first order in ϕ (given values for G, M, and R, why can we take $\phi^2 \ll 1$?). Since these are non-relativistic particles, you may assume, at the appropriate point, that $|U^t| \gg |U^z|$, and thus that $|U^t|^2 \approx -1$.

If we had used a different metric to describe the same newtonian spacetime, rather than that in Eq. (i), would we have obtained a different result for the geodesic deviation, Eq. (iii)? Explain your answer. [d^+u^+]

4

Energy, Momentum, and Einstein's Equations

Proving that nothing ever changes:

> … For Aristotle divides theoretical philosophy too, very fittingly, into three primary categories, physics, mathematics and theology. … Now the first cause of the first motion of the universe, if one considers it simply, can be thought of as an invisible and motionless deity; the division [of theoretical philosophy] concerned with investigating this [can be called] 'theology', since this kind of activity, somewhere up in the highest reaches of the universe, can only be imagined, and is completely separated from perceptible reality. The division which investigates material and ever-moving nature, and which concerns itself with 'white', 'hot', 'sweet', 'soft' and suchlike qualities one may call 'physics'; such an order of being is situated (for the most part) amongst corruptible bodies and below the lunar sphere. That division which determines the nature involved in forms and motion from place to place, and which serves to investigate shape, number, size, and place, time and suchlike, one may define as 'mathematics' …
>
> From all this we conclude: that the first two divisions of theoretical philosophy should rather be called guesswork than knowledge, theology because of its completely invisible and ungraspable nature, physics because of the unstable and unclear nature of matter; hence there is no hope that philosophers will ever be agreed about them; and that only mathematics can provide sure and unshakeable knowledge to its devotees, provided one approaches it rigorously. For its kind of proof proceeds by indisputable methods, namely arithmetic and geometry. … As for physics, mathematics can make a significant contribution. For almost every peculiar attribute of material nature becomes apparent from the peculiarities of its motion from place to place.
>
> *Preface to Book 1 of Ptolemy's* Almagest*, between 150–161* CE

Ptolemy is right, here (though some of the details of his cosmology have been adjusted since he wrote this, and what he refers to as 'theology' is now more often referred to as 'Quantum Gravity'): mathematics we can know all about, with certainty; for physics we have to make guesses. He is also correct about the contribution of mathematics, and we'll discover that our first insights in this section do indeed come from considering the peculiarities of the material world's motion from place to place.

To match our return to physics, here, we're now going to specialise to working in a four-dimensional manifold, and to metrics with a signature of $+2$, so that the LIF has the same metric as Minkowski space. To further match Section 2.3.4, we will also introduce a slight, and traditional, notational change. We will now index components with greek letters, μ, ν, α, β, ..., which we take to run from 0 to 3; we will sometimes write indexes with latin letters i, j,..., taking these to run over the spacelike directions, 1, 2, and 3.

4.1 The Energy-Momentum Tensor

The point of this whole book is to describe how gravity, in the form of the curvature of space-time, is determined by the presence of mass. In newtonian physics, the relationship is straightforward, since the notion of mass is unproblematic. In relativity, however, we know that what matters is not mass alone, but energy-momentum, and so it is not unreasonable that what matters in GR is not mass, but the distribution of energy-momentum, and so we must find a way of describing this in an acceptably geometrical fashion. In this section we are confining ourselves to *special relativity (SR)*, and in the next section we discover that this is not, physically, a restriction in fact. This section largely follows Schutz chapter 4. MTW (1973, chapter 5) takes a significantly different approach, but is very illuminating.

We start (as we all end) with dust.

4.1.1 Dust, Fluid, and Flux

A *fluid* in GR and cosmology is, not surprisingly, something that flows; that is, a substance where the forces perpendicular to an imaginary surface (i.e., pressure) are much greater than the forces parallel to it (i.e., stress, arising from viscosity). The limit of this, a *perfect fluid*, is a substance that has pressure but zero stresses. An even simpler substance is termed *dust*, which denotes an idealised form of matter, consisting of a collection of non-interacting particles

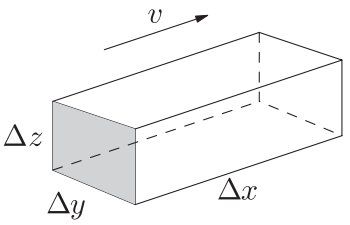

Figure 4.1 The volume swept out by an area.

that are not moving relative to each other, so that the collection has zero pressure – the dust's only physical property is mass-density. That is to say that there is a frame, called the *momentarily comoving reference frame-* (MCRF), with respect to which all the particles in a given volume have zero velocity.[1]

We can suppose for the moment that all the dust particles have the same (rest) mass m, but that different parts of the dust cloud may have different number densities n. Just as the particle mass m is the mass in the particle's rest frame, the number density n is always that measured in the MCRF.

If we Lorentz-transform to a frame that is moving with velocity $\bar{v}$ with respect to the MCRF, a (stationary) volume element of size $\Delta x \Delta y \Delta z$ will be Lorentz-contracted into a (moving) element of size $\Delta x' \Delta y' \Delta z' = (\Delta x/\gamma)\Delta y \Delta z$, where γ is the familiar Lorentz factor $\gamma = (1 - v^2)^{-1/2}$, supposing that the frames are chosen such that the relative motion is along the x-axis. That means that the number density of particles, as measured in the frame relative to which the dust is moving, goes up to γn. What, then, is the *flux* of particles through an area $\Delta y \Delta z$ in the y'–z' plane? The particles in the volume all pass through the area $\Delta y' \Delta z'$ in a time $\Delta t'$, where $\Delta x' = v \Delta t'$, and so this total number of particles is $(\gamma n)(v \Delta t')\Delta y' \Delta z'$. Thus the total number of particles per unit time and per unit area, which is the flux in the x'-direction, is $\gamma n v$. Writing N^x for this x-directed flux, and v^x for the velocity along the x-axis, v, this is

$$N^x = \gamma n v^x. \tag{4.1}$$

We can generalise this, and guess that we can reasonably define a flux vector

$$\overline{N} = n\overline{U}, \tag{4.2}$$

where again n is the dust number density in its MCRF, and $\overline{U}$ is the 4-velocity vector $(\gamma, \gamma v^x, \gamma v^y, \gamma v^z)$. Since the velocity vector has the property $\mathsf{g}(\overline{U}, \overline{U}) = -1$ (remember your SR, and that the 4-velocity vector $\overline{U} = (1, \overline{0})$

[1] This is also, interchangeably, sometimes called the Instantaneously Comoving Reference Frame (ICRF).

in MCRF), we have $\mathsf{g}(\overline{N},\overline{N}) = N_\alpha N^\alpha = -n^2$. The components of the flux vector $\overline{N}$ in this frame are

$$(\gamma n, \gamma n v^x, \gamma n v^y, \gamma n v^z). \tag{4.3}$$

This flux vector is a *geometrical* object, because $\overline{U}$ is, and so although its components are frame-dependent, the vector as a whole is not.

It is obvious how to recover, from Eq. (4.3), the fluxes N^x across surfaces of constant coordinate – we simply take the components – but we will need to be more general than this. Any function defined over space-time, $\phi(t,x,y,z)$, defines a surface $\phi =$ constant, and its gradient one-form $\widetilde{\mathrm{d}}\phi$ acts as a normal to this surface (think of the planes in our visualisation of one-forms). The *unit* gradient one-form $\widetilde{n} \equiv \widetilde{\mathrm{d}}\phi / |\widetilde{\mathrm{d}}\phi|$ points in the same direction but has unit magnitude (the notational clash with the number density n is unfortunate but conventional). Consider specifically the coordinate function x: the gradient one-form corresponding to this, $\widetilde{\mathrm{d}}x$, has components $(0,1,0,0)$ (and so is already a unit one-form). If we contract this one-form with the flux vector, we find

$$\langle \widetilde{\mathrm{d}}x, \overline{N} \rangle \equiv \overline{N}(\widetilde{\mathrm{d}}x) = N^x$$

(where the last expression denotes the x-component of $\overline{N}$, rather than the whole set of components). That is, contracting the flux vector with a gradient one-form produces the flux across the corresponding surface; this is true in general, so that $\overline{N}(\widetilde{n})$ produces the flux across the surface $\phi =$ constant, where $\widetilde{n} = \widetilde{\mathrm{d}}\phi / |\widetilde{\mathrm{d}}\phi|$. The vector $\overline{N} = n\overline{U}$ is manifestly geometrical; it is our ability to recover the flux in this way that justifies our naming this the 'flux vector'.

4.1.2 The Energy-Momentum Tensor

We'll switch from (t,x,y,z) to general (x^0,x^1,x^2,x^3), now, but we're still confining ourselves to SR.

We know from our study of SR that energy and mass are interconvertible. For our dust particles of mass m, therefore, the *energy density* of the dust, in the MCRF, is mn. In our moving frame, however, as well as the number density rising to γn, the total energy of each particle, as measured in the 'stationary frame', goes up to γm. Thus the energy density of the dust as measured in a moving frame is $\gamma^2 mn$. This double factor of γ cannot result from a Lorentz boost of a vector, and is the first indication that to describe the energy-momentum of the dust we will need to use a higher-order tensor.

What geometrical objects do we have to play with? We have the momenta of the dust particles, $\overline{p} = m\overline{U}$, and we have the flux vector $\overline{N} = n\overline{U}$. As mentioned

in the previous section, we also have the gradient one-forms corresponding to the coordinate functions, $\tilde{\mathrm{d}}x^\alpha$. By contracting the vectors with these one-forms we can extract the particles' energy $p^0 = \overline{p}(\tilde{\mathrm{d}}x^0)$ or spatial momenta $p^i = \overline{p}(\tilde{\mathrm{d}}x^i)$, or the number density $N^0 = \overline{N}(\tilde{\mathrm{d}}x^0)$ (which we can interpret as the number crossing a surface of constant time, into the future) or number flux $N^i = \overline{N}(\tilde{\mathrm{d}}x^i)$.

Let us form the $\binom{2}{0}$ tensor

$$\mathsf{T} = \overline{p} \otimes \overline{N} = \rho \overline{U} \otimes \overline{U}, \qquad \text{dust} \tag{4.4}$$

(writing $\rho = mn$ for the *mass density*, and recalling the definition of outer product in Section 2.2.2) – this is known as both the *energy-momentum tensor* and the stress-energy tensor. In order to convince ourselves that this mathematical object has a useful physical interpretation, we now examine the components of this tensor, obtained by contracting it with the basis one-forms $\tilde{\omega}^\alpha = \tilde{\mathrm{d}}x^\alpha$, where the coordinate functions x^i are those corresponding to a frame with respect to which the dust is moving. These components are, of course,

$$T^{\alpha\beta} = \mathsf{T}(\tilde{\mathrm{d}}x^\alpha, \tilde{\mathrm{d}}x^\beta) = \overline{p}(\tilde{\mathrm{d}}x^\alpha) \times \overline{N}(\tilde{\mathrm{d}}x^\beta).$$

The 0-0 component T^{00} is just $\gamma^2 mn$, which we can recognise as the energy density of the dust, or the flow of the zeroth component of momentum across a surface of constant time.

The 0-i component is $T^{0i} = \gamma m \times \gamma n v^i$ (after comparing with Eq. (4.3)). Given that nv has the dimensions of (per-unit-area per-unit-time), and that mass and energy are interconvertible in relativity, this is identifiable as the flux of energy across a (spatial) surface of constant x^i.

The i-0 component $T^{i0} = p^i \times N^0 = m\gamma v^i \times \gamma n$ is the flux of the ith component of momentum across a surface of constant time, into the future. By analogy with the energy density, this is known as (the ith component of) the *momentum density* of the dust. Now, energy flux across a surface is an amount of energy-per-unit-time, per unit area or, since energy and mass are the same thing, mass-per-unit-time, per unit area. However, momentum density is the amount of momentum per unit volume, which is mass-times-speed per unit volume, which is dimensionally the same as energy flux. Another way of getting to the same place (in Schutz's words this time) is that energy flux is the density of mass-energy times the speed it flows at, whereas momentum density is the mass density times the speed the mass flows at, which is the same thing. Thus the identity of T^{i0} and T^{0i} in this case is not coincidental or special to dust, but quite general:

$$T^{i0} = T^{0i}.$$

Finally the *i*-*j* component of the energy-momentum tensor, $T^{ij} = p^i N^j = \gamma m v^i \times \gamma n v^j$, is the flux of i-momentum across a surface of constant x^j. It has the dimensions of momemtum per unit time, per unit area, leading us to identify it as force per unit area, or pressure.

In general, therefore, we can interpret the component $\mathsf{T}(\tilde{\mathrm{d}}x^\alpha, \tilde{\mathrm{d}}x^\beta)$ as the flow of the αth component of momentum across a surface of constant coordinate x^β.

By considering the torques acting on a fluid element we can show (Schutz §4.5) that the tensor T is symmetric in general,

$$T^{\alpha\beta} = T^{\beta\alpha}, \qquad \text{or } \mathsf{T}(\tilde{p}, \tilde{q}) = \mathsf{T}(\tilde{q}, \tilde{p}), \quad \forall \tilde{p}, \tilde{q}. \tag{4.5}$$

In a perfect fluid, there is no preferred direction, so the spatial part of the energy-momentum tensor must be proportional to the spatial part of the metric, which is δ^{ij} in SR. Since there is no viscosity, the only momentum transport possible is perpendicular to the surface of a fluid element, in the form of pressure p (which is force per unit area, remember), giving the constant of proportionality (see Schutz §4.6 for an expanded version of this argument), and so

$$T^{ij} = p\delta^{ij}, \qquad \text{(perfect fluid)}. \tag{4.6}$$

From there it is a short step to show that the energy-momentum tensor for a perfect fluid, as a geometrical object, is

$$\mathsf{T} = (\rho + p)\overline{U} \otimes \overline{U} + p\mathsf{g}, \qquad \text{(perfect fluid)}. \tag{4.7}$$

Dust has no pressure, so its energy-momentum tensor in the MCRF is

$$\mathsf{T} = \mathrm{diag}(\rho, 0, 0, 0), \qquad \text{(dust, MCRF)}. \tag{4.8}$$

The final important property of this tensor is its conservation law. If energy is to be conserved, then the amount of energy-momentum entering an arbitrary four-dimensional box must be the same as the amount leaving it. From this we promptly deduce that

$$\frac{\partial}{\partial x^0}T^{\alpha 0} + \frac{\partial}{\partial x^1}T^{\alpha 1} + \frac{\partial}{\partial x^2}T^{\alpha 2} + \frac{\partial}{\partial x^3}T^{\alpha 3} = 0,$$

or

$$T^{\alpha\beta}{}_{,\beta} = 0. \tag{4.9}$$

By a similar sort of argument, requiring that under any flow of a fluid or of dust the total number of particles is unchanged, we can show that

$$N^\alpha{}_{,\alpha} = (nU^\alpha)_{,\alpha} = 0, \tag{4.10}$$

with no source term on the right-hand side. [Exercise 4.1]

4.1.3 The Energy-Momentum Tensor: A Different Approach

Imagine a wireframe box marking out a cuboidal volume of space.[2] It might have just air in it, or flowing through it if you're sitting in a draught or throwing it from hand to hand. You might take it to a tap and let water run through the volume it marks out (you could also imagine putting it round a candle flame and watching exothermic chemistry happen in it, or waving a magnet near it and measuring electromagnetism happening inside it; but we'll stick to particle and continuum mechanics for the moment).

How much energy is there inside the wire box? Or, if it's moving, how much dynamics – how much *oomph* – is moving through the box? If there's a particle inside the box then we can talk about its energy-momentum $\overline{p} = m\overline{U} = m\gamma(1, \mathbf{v})$; the components of this quantity are of course frame-dependent, taking different values in frames attached to you, to the box, or to the particles within it, but we know that the 4-momentum of $\overline{p}$ is a geometrical frame-independent quantity. If there are two particles inside the box, we can simply add up their momenta, but if there is a continuum of dynamics – such as the air flowing through the box – then we must talk of the *density* of energy-momentum; but for that we must first find a way of talking about a volume in a frame-independent way.

Define the 1-form $\widetilde{\sigma}$ with components

$$\sigma_\mu = \epsilon_{\mu\alpha\beta\gamma} A^\alpha B^\beta C^\gamma, \tag{4.11}$$

where $\overline{A} = (A^0, \mathbf{a})$, $\overline{B} = (B^0, \mathbf{b})$, and $\overline{C} = (C^0, \mathbf{c})$ are three linearly independent vectors, and $\epsilon_{\mu\alpha\beta\gamma}$ is the *Levi-Civita symbol*, which is such that

$$\epsilon_{\mu\alpha\beta\gamma} = \begin{cases} +1 & \text{if } \mu\alpha\beta\gamma \text{ is an even permutation of 0123} \\ -1 & \text{if it is an odd permutation} \\ 0 & \text{otherwise.} \end{cases} \tag{4.12}$$

If $\overline{A}$, $\overline{B}$, and $\overline{C}$ are purely spacelike vectors, so that $A^0 = B^0 = C^0 = 0$, then $\sigma_i = 0$ and

$$\sigma_0 = \epsilon_{0\alpha\beta\gamma} A^\alpha B^\beta C^\gamma,$$

and you may recall from linear algebra that this is the expression for $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$, the vector triple product, which gives the volume, V, of the parallelepiped bounded by the three 3-vectors **a**, **b**, and **c**. Since each of these vectors is

[2] The account here is closely compatible with MTW (1973, chaper 5). It covers the same material as the previous section, but in a less heuristic way, at the expense of introducing some new maths. It doesn't have a 'dangerous bend' marker but you should feel free to skip it if the previous section was adequately satisfying to your mathematical sensibilities.

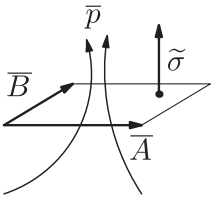

Figure 4.2 A volume form (dimension $\overline{C}$ suppressed).

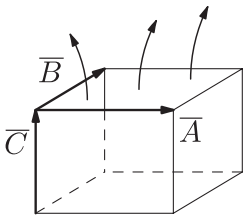

Figure 4.3 Flux out of a box.

orthogonal to $\widetilde{\sigma}$ (that is, $\langle \widetilde{\sigma}, \overline{A} \rangle = 0$, etc.), the 3-d volume that they span is a hyperplane of the one-form $\widetilde{\sigma}$. This volume (shown in Figure 4.2, with the direction $\overline{C}$ suppressed) contains matter or other energy with associated momenta $\overline{p}$.

This (timelike) volume is moving through space-time with a velocity $\overline{U}$, where $\overline{U} = (1, \mathbf{0})$ in the volume's rest frame. The one-form dual to this velocity is $\widetilde{U} = \mathsf{g}(\overline{U}, \tilde{\cdot})$, which has components $\widetilde{U} = (-1, \mathbf{0})$. Note that, promptly from this definition,

$$\langle \widetilde{U}, \overline{U} \rangle = \mathsf{g}(\overline{U}, \overline{U}) = -1.$$

Notice also that, as constructed in this section, with $\sigma_0 = V$ and $\sigma_i = 0$, the volume one-form $\widetilde{\sigma}$ has the same components as $-V\widetilde{U}$ in this frame, and therefore in *any* frame $\widetilde{\sigma} = -V\widetilde{U}$. With this identification, it is natural to interpret $\widetilde{\sigma}$ as representing the volume moving into the future, along its rest frame's t-axis.

Now picture the volume bounded by $\overline{A}$, $\overline{B}$, and $\overline{C}$ as a box whose top, in the plane $\overline{A}\,\overline{B}$, is opened, allowing its contents to puff out into the surrounding space (for simplicity, take **a**, **b**, and **c** to be orthogonal and of length L, so $\mathbf{a} = (L, 0, 0)$, $\mathbf{b} = (0, L, 0)$, and $\mathbf{c} = (0, 0, L)$). In a time $\Delta\tau$, this box lid moves through time by a displacement $\overline{T} = (\Delta\tau, \mathbf{0})$. If we now calculate $\sigma_\mu = \epsilon_{\mu\alpha\beta\gamma} A^\alpha B^\beta T^\gamma$, we find $\sigma_3 = L^2 \Delta\tau$, with the other components zero. This (spacelike) volume, which is parallel to $\widetilde{C}$ (and thus in the direction of the basis one-form $\tilde{\omega}^3$), clearly represents the amount of space-time swept out by the top of the box, $\overline{A}\,\overline{B}$, in time $\Delta\tau$. The one-form $\widetilde{\sigma}$, therefore, represents a

volume of space-time, at a particular point in space-time, which is spacelike or timelike depending on its orientation.

This volume one-form gives us a way of answering the question 'how much energy-momentum is inside the wireframe box?' We can define a $\binom{2}{0}$ tensor T which (you will not at this point be surprised to learn) we can call the 'energy-momentum tensor', the action of which, when given a volume one-form $\widetilde{\sigma}$, is to produce the quantity of energy-momentum contained within that volume:

$$\overline{p}_{\text{box}} = \mathsf{T}(\tilde{\cdot}, \widetilde{\sigma}). \tag{4.13}$$

We can see that this makes sense, in a number of ways, using the examples of $\widetilde{\sigma}$ mentioned previously.

Contracting T with the timelike $\widetilde{\sigma}$, in Eq. (4.13), gives us the 4-momentum contained within that (spacelike) volume, and directed into the future. The energy instantaneously contained within the box, which is the zeroth component of $\overline{p}_{\text{box}}$ in its rest frame, can be extracted by contracting it with the basis one-form $\widetilde{\omega}^0$ which, in this frame, is just $-\widetilde{U}$, to give

$$E = \overline{p}_{\text{box}}(\widetilde{\omega}^0) = -\overline{p}_{\text{box}}(\widetilde{U}) = \mathsf{T}(\widetilde{U}, \widetilde{\sigma}) = +V\mathsf{T}(\widetilde{U}, \widetilde{U}),$$

which allows us to identify $\mathsf{T}(\widetilde{U}, \widetilde{U})$ as the energy density within the volume (moving into the future), in the box's rest frame; similarly $p^i = -V(\widetilde{\omega}^i, \widetilde{U})$ is the momentum density within the box.

Looking now at the spacelike $\widetilde{\sigma}$, we find that $\overline{p}_{\text{box}} = L^2 \Delta\tau \mathsf{T}(\tilde{\cdot}, \widetilde{\omega}^3)$, so that we can identify $\mathsf{T}(\tilde{\cdot}, \widetilde{\omega}^3)$ as a flux of momentum through the top of the $\overline{A}\,\overline{B}\,\overline{C}$ box.

Returning to the dust, we can recall the number density N^0 and flux N^i of dust particles, and thus obtain the density and flux of energy-momentum $p^\mu N^0$ and $p^\mu N^i$, but these are the components of a tensor: $p^\mu N^\nu = \mathsf{S}(\widetilde{\omega}^\mu, \widetilde{\omega}^\nu)$, where $\mathsf{S} = \overline{p} \otimes \overline{N}$. The tensor S is therefore the energy-momentum tensor for this dust, as defined by Eq. (4.13), and comparison with Eq. (4.4) shows that this tensor is exactly the energy-momentum tensor we obtained earlier.

The expression for the volume form, Eq. (4.11), may appear to be pulled from a hat, but in fact it emerges fairly naturally from a larger theory of *differential forms*; the one-forms we have been using are the simplest objects in a sequence of n-forms. This 'exterior calculus' can be used, amongst other things, for providing yet another way of discussing differentiation (and integration) in a curved manifold, alongside the covariant derivative we have extensively used, and the Lie derivative we have mentioned in passing. MTW discuss this approach in their chapter 4; Carroll describes them, very lucidly, in his section 2.9; Schutz (1980) gives an extensive treatment in chapter 4; they are well covered in other advanced GR textbooks.

[Exercise 4.2]

4.1.4 Maxwell's Equations

For completeness, here are Maxwell's equations in the form appropriate for GR. For fuller details, see exercise 25 in Schutz's §4.10.

Given electromagnetic fields (E^x, E^y, E^z) and (B^x, B^y, B^z), we can define the antisymmetric *Faraday tensor*

$$\mathsf{F} = \begin{pmatrix} 0 & E^x & E^y & E^z \\ -E^x & 0 & B^z & -B^y \\ -E^y & -B^z & 0 & B^x \\ -E^z & B^y & -B^x & 0 \end{pmatrix}. \tag{4.14}$$

We can also define the current vector $\overline{J} = (\rho, j^x, j^y, j^z)$ corresponding to a charge density ρ and current 3-vector $\mathbf{j}$. With these definitions, Maxwell's equations in SR become

$$F^{\mu\nu}{}_{,\nu} = 4\pi J^\mu \tag{4.15a}$$

$$F_{\mu\nu,\lambda} + F_{\nu\lambda,\mu} + F_{\lambda\mu,\nu} = 0. \tag{4.15b}$$

The Faraday tensor F and the energy-momentum tensor T together form the source for the gravitational field. Notwithstanding that, we shall not explicitly include the Faraday tensor in the discussion that follows.

The Faraday tensor, and Maxwell's equations, take a particularly compact and elegant form when expressed in terms of the exterior derivatives mentioned in passing at the end of Section 4.1.3.

It is possibly worth highlighting that the components of Eq. (4.14) are manifestly frame-dependent – you can pick a frame where either $\overline{E}$ or $\overline{B}$ disappears:

> It is known that Maxwell's electrodynamics – as usually understood at the present time – when applied to moving bodies, leads to asymmetries which do not appear to be inherent in the phenomena. . . . Examples of this sort, together with the unsuccessful attempts to discover any motion of the earth relatively to the "light medium," suggest that the phenomena of electrodynamics as well as of mechanics possess no properties corresponding to the idea of absolute rest.
>
> *(Einstein, 1905, paragraphs 1and 2).*

4.2 The Laws of Physics in Curved Space-time

So we now have a way to describe the energy-momentum contained within an arbitrary distribution of matter and electromagnetic fields. What we now want to know is how these relate to the curvature of the space-time they lie within.

4.2.1 Ricci, Bianchi, and Einstein

First we need to establish useful contractions of the curvature tensor. See Schutz §6.6 for further details of this brief relapse into mathematics.

These contractions are the *Ricci tensor*, obtained by contracting the full curvature tensor over its first and third indexes,

$$R_{\beta\nu} \equiv g^{\alpha\mu} R_{\alpha\beta\mu\nu} = R^{\mu}{}_{\beta\mu\nu}, \tag{4.16}$$

and the *Ricci scalar* obtained by further contracting the Ricci tensor,

$$R \equiv g^{\beta\nu} R_{\beta\nu} = g^{\beta\nu} g^{\alpha\mu} R_{\alpha\beta\mu\nu}. \tag{4.17}$$

Note, from Eq. (3.55a), that the Ricci tensor is symmetric: $R_{\alpha\beta} = R_{\beta\alpha}$.

By differentiating Eq. (3.54), we can find

$$2R_{\alpha\beta\mu\nu,\lambda} = g_{\alpha\nu,\beta\mu\lambda} - g_{\alpha\mu,\beta\nu\lambda} + g_{\beta\mu,\alpha\nu\lambda} - g_{\beta\nu,\alpha\mu\lambda}, \tag{4.18}$$

and noting that partial derivatives commute, deduce

$$R_{\alpha\beta\mu\nu,\lambda} + R_{\alpha\beta\lambda\mu,\nu} + R_{\alpha\beta\nu\lambda,\mu} = 0. \tag{4.19}$$

Recall that Eq. (3.54) was evaluated in LIF coordinates; however, since in these coordinates $\Gamma^{\mu}_{\alpha\beta} = 0$ (though $\Gamma^{\mu}_{\alpha\beta,\sigma}$ need not be zero), partial differentiation and covariant differentiation are equivalent, and Eq. (4.19) can be rewritten

$$R_{\alpha\beta\mu\nu;\lambda} + R_{\alpha\beta\lambda\mu;\nu} + R_{\alpha\beta\nu\lambda;\mu} = 0, \tag{4.20}$$

which is a tensor equation, known as the *Bianchi identities*.

If we perform the Ricci contraction of Eq. (4.16) on the Bianchi identities, we obtain

$$R_{\beta\nu;\lambda} - R_{\beta\lambda;\nu} + R^{\mu}{}_{\beta\nu\lambda;\mu} = 0, \tag{4.21}$$

and if we contract this in turn, we find the *contracted Bianchi identity*

$$G^{\alpha\beta}{}_{;\beta} = 0, \tag{4.22}$$

where the (symmetric) *Einstein tensor* G is defined as

$$G^{\alpha\beta} \equiv R^{\alpha\beta} - \tfrac{1}{2} g^{\alpha\beta} R. \tag{4.23}$$

From its name, and the alluring property Eq. (4.22), you can guess that this tensor turns out to be particularly important for us. There are some further remarks about the Ricci tensor in Section 4.2.5. Some texts define the Einstein and Ricci tensors with a different overall sign; see Section 1.4.3.

Anyway, back to the physics.

4.2.2 The Equivalence Principle

Back in Chapter 1, we first described the equivalence principle (EP). It is now finally time to use this, and to restate it in terms that take advantage of the mathematical work we have done. The material in this section is well-discussed in Schutz §7.1, as well as in Rindler (2006), at the end of his chapter 1 and in §§8.9–8.10. It is discussed, one way or another, in essentially every GR textbook, with more or less insight, so you can really take your pick.

One statement of the principle (Einstein's, in fact) is

> *The Equivalence Principle (EP):* All local, freely falling, nonrotating laboratories are fully equivalent for the performance of all physical experiments.

Rindler refers to this as the 'strong' EP, and discusses it under that title with characteristic care, distinguishing it from the 'semistrong' EP, and the 'weak' EP, which is the statement that inertial and gravitational mass are the same.

The EP gives us a route from the physics we understand to the physics we don't (yet). That is, given that we understand how to do physics in the inertial frames of SR, we can import this understanding into the apparently very different world of curved – possibly completely round the twist – space-times, since the EP tells us that physics works locally in *exactly* the same way in *any* LIF, free-falling in a curved space-time.

So that tells us that an electric motor, say, will work as happily as we free-fall into a black hole, as it would work in any less doomed SR frame. It does immediately constrain the general form of physical laws, since it requires that, whatever their form in general, they must reduce to the SR version when expressed in the coordinates of a LIF. For example, whatever form Maxwell's equations take in a curved space-time, they must reduce to the SR form, Eq. (4.15), when expressed in the coordinates of any LIF. The same goes for conservation laws such as Eq. (4.9) or Eq. (4.10). This form of the EP doesn't, however, rule out the possibility that the curved space-time law is (much) more complicated in general, and simply (and even magically) reduces to a simple SR form when in a LIF. Specifically, it doesn't rule out the possibility of *curvature coupling*, where the general form of a conservation law such as Eq. (4.9) has some dependence on the local curvature, which disappears in a LIF.

For that, we need a slightly stronger wording of the EP as quoted earlier in the section (see Schutz §7.1; Rindler §8.9 quotes this as a 'reformulation' of the EP):

> *The Strong Equivalence Principle:* Any physical law that can be expressed in tensor notation in SR has exactly the same form in a locally inertial frame of a curved space-time. (4.24)

The difference here is that this says, in effect, that only geometrical statements count (this is why we've been making such a fuss about the primacy of geometrical objects, and the relative unimportance of their components, throughout the book). That is, it says that a SR conservation law such as Eq. (4.9), $T^{\mu\nu}{}_{,\nu} = 0$, has the same *form* in a LIF, and as a result, because covariant differentiation reduces to partial differentiation in the LIF, the partial derivative here is really just the LIF form of a covariant derivative, and so the general form of this law is

$$T^{\mu\nu}{}_{;\nu} = 0, \tag{4.25}$$

with the comma turning straight into a semicolon, and *no extra curvature terms appearing on the right-hand side*. That is why this form of the EP is sometimes referred to as the '*comma-goes-to-semicolon*' rule.

Note that this comma-goes-to-semicolon is emphatically *not* what happened in the step between, for example, Eq. (4.19) and Eq. (4.20), and in various similar manouvres throughout Chapter 3 (such as before Eq. (3.41) and after Eq. (3.54)). What was happening there was a *mathematical* step: covariant differentiation of a geometrical object is equivalent to partial differentiation when in a LIF. We have a true statement about partial differentiation in Eq. (4.19), so the same statement must be true of covariant differentiation; such a statement in one frame is true in any frame, hence the generality. The Strong EP comma-goes-to-semicolon rule, on the other hand, is making a *physical* statement, namely that the statement of a physical law in a LIF directly implies a fully covariant law, which is *no more complicated*.

It is possibly not obvious, but the Strong EP also tells us how matter is affected by space-time. In SR, a particle at rest in an inertial frame moves along the time axis of the Minkowski diagram – that is, along the timelike coordinate direction of the LIF, which is a geodesic. The Strong EP tells us that the same must be true in GR, so that this picks out the curves generated by the timelike coordinate of a LIF, which is to say:

> *Space tells matter how to move:* Free-falling particles move on timelike geodesics of the local space-time. (4.26)

This, like the Strong EP, is a physical statement about our universe, rather than a mathematical one. 'We will return to this very important point in the sections to follow.' [Exercises 4.3 and 4.4]

4.2.3 Geodesics and the Link to 'Gravity'

We should say a little more about the rather bald statement (4.26).

This statement describes the motion of a particle in a particular space-time. If you want to describe or predict the motion of a particle, you do it in two

steps. First, you work out which geodesic it will travel along: this involves solving Einstein's equations, and working out from the initial conditions of the motion which geodesic your particle is actually on, amongst the large number of possible geodesics going through the initial point in space-time. Secondly, you work out how to translate from the simple motion in the inertial coordinates attached to the particle, to the coordinates of interest (presumably attached to you).

The key thing on the way to the important insight here, is to note that if you're moving along a geodesic – if you're in free fall – *you are not being accelerated* , in the very practical sense that if you were carrying an accelerometer, it would register no acceleration. If instead you stand still on earth, and drop a ball from your hand, the ball is showing the path you would have taken, were it not for the floor. That is, it is the force exerted by the floor on your feet that is accelerating you away from your 'natural' free-fall path. If you hold an accelerometer in your hand – for example, a mass on a spring – you can see your acceleration register as the spring extends beyond the length it would have in free fall.

In other words, we've been thinking of this situation backwards. We're used to standing on the ground being the normal state, and falling being the exceptional one (we're primates, after all, and not falling out of trees has long been regarded as a key skill). But GR says that we've got that inside out: inertial motion, which in the presence of masses we recognise as free fall, is the simplest, or normal, or 'natural' motion, requiring no explanation, and it is not-falling that has to be explained.[3] The EP says that the force of gravity doesn't just *feel* like being forced upwards by the floor, it *is* being accelerated upwards by the floor.

4.2.4 Einstein's Equations

We have, in the run-up to statement (4.26), worked out how space-time affects the motion of matter. We now have to work out how matter affects space-time – where does 'gravity' come from? We can't *deduce* this from anywhere; we can simply make intelligent guesses about it, based on our experience of other parts of physics – see Ptolemy's remarks about this at the beginning of this chapter – and hope that our (mathematical) deductions from these are corroborated, or not, by experiment. Thus our goal in this section is to *make*

[3] This term 'natural motion' is clearly not being used in a technical sense. The history of physics might be said to consist of a sequence of attempts – by Aristotle, Ptolemy, Kepler, Galileo, Newton, and Einstein – to identify a successively refined idea of 'natural motion' which adequately and fundamentally explains the observed behaviour of the cosmos. Currently 'move along your locally-minkowskian *t*-axis' is it.

Einstein's equations plausible . Schutz does this in his §§8.1–8.2; Rindler does it very well in his §§8.2 and 8.10; essentially every textbook on GR does it in one way or another, either heuristically or axiomatically.

Newton's theory of gravity can be expressed in terms of a gravitational field ϕ. The gravitational force $\mathbf{f}$ on a test particle of mass m is a three-vector with components $f_i = -m\phi_{,i}$, and the source of the field is mass density ρ, with the field equation connecting the two being

$$\phi^{,i}{}_{,i} = 4\pi G\rho \tag{4.27}$$

(with the sum being taken over the three space indexes, and where $\phi^{,i}{}_{,i} = g^{ij}\phi_{,ij} = g^{ij}\partial^2\phi/\partial x^i\partial x^j$). This is Poisson's equation. In a region that does not contain any matter – for example an area of space that is not inside a star or a planet or a person – the mass density $\rho = 0$, and the *vacuum* field equations are

$$\phi^{,i}{}_{,i} = 0. \tag{4.28}$$

Now cast your mind back to Chapter 1, and the expression in the notes there for the acceleration towards each other of two free-falling particles. This expression can be slightly generalised and rewritten here as

$$\frac{\mathrm{d}^2\xi^i}{\mathrm{d}t^2} = -\phi^{,i}{}_{,j}\xi^j. \tag{4.29}$$

But compare this with Eq. (3.60): they are both equations of geodesic deviation, suggesting that the tensor represented by $R^\mu{}_{\alpha\nu\beta}U^\alpha U^\beta$ is analogous to $\phi^{,i}{}_{,j}$ (we've used the symmetries of the curvature tensor to swap two indexes, note, and used $\overline{U}$ rather than $\overline{X}$ to refer to the free-falling particle velocity). Since the particle velocities are arbitrary, that means, in turn, that the $\phi^{,i}{}_{,i}$ appearing in Poisson's equation is analogous to $R_{\alpha\beta} = R^\mu{}_{\alpha\mu\beta}$, and so a good guess at the relativistic analogue of Eq. (4.28) is

$$R_{\mu\nu} = 0. \tag{4.30}$$

This guess turns out to have ample physical support, and Eq. (4.30) is known as Einstein's *vacuum* field equation for GR.

If $R_{\mu\nu} = 0$, then $R = g^{\mu\nu}R_{\mu\nu} = 0$ and therefore

$$G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu} = 0.$$

So much for the vacuum equations, but we want to know how space-time is affected by matter. We can't relate it simply to ρ, since Section 4.1.2 made it clear that this was a frame-dependent quantity; the field is much more likely to be somehow bound to the E-M tensor T instead. Looking back at Eq. (4.27), we might guess

$$R^{\mu\nu} = \kappa T^{\mu\nu} \tag{4.31}$$

as the field equations in the presence of matter, where κ is some coupling constant, analogous to the newtonian gravitational constant G. This looks plausible, but the conservation law Eq. (4.25) immediately implies that ${R^{\mu\nu}}_{;\nu} = 0$, which, using the Bianchi identity Eq. (4.22), in turn implies that $R_{;\nu} = 0$. But if we use Eq. (4.31) again, this means that $(g_{\alpha\beta}T^{\alpha\beta})_{;\nu} = 0$ also. If we look back to, for example, Eq. (4.8), we see that this field equation, Eq. (4.31), would imply that the universe has a constant density. Which is not the case. So Eq. (4.31) cannot be true.[4]

So how about

$$G^{\mu\nu} = \kappa T^{\mu\nu} \tag{4.32}$$

as an alternative? The Bianchi identity Eq. (4.22) tells us that the conservation equation ${T^{\mu\nu}}_{;\nu} = 0$ is satisfied identically. Additionally – and this is the key part of the argument – numerous experiments tell us that Eq. (4.32) has so-far undisputed physical validity: it has not been shown to be incompatible with our universe. It is known as the *Einstein field equation* , and allows us to complete the other half of the famous slogan

> Space tells matter how to move – the statement (4.26) plus equations (3.42) or (3.43). And matter tells space how to curve – equation (4.32).

Einstein first published these equations in a series of papers delivered to the Prussian Academy of Sciences in November 1915; there is a detailed account of Einstein's actual sequence of ideas, which is slightly (but, remarkably, only slightly) more tentative than the description in this section may suggest, in Janssen and Renn (2015).

There are two further points to make, both relating to the arbitrariness that is evident in our justification of Eq. (4.32).

The first is to acknowledge that, although we were forced to go from Eq. (4.31) to Eq. (4.32) by the observation that the universe is in fact lumpy, there is nothing other than Occam's razor that forces us to stop adding complication when we arrive at Einstein's equations. There have been various attempts to play with more elaborate theories of gravity, but almost none so far that have acquired experimental support. Chandrasekhar's words on this, quoted in Schutz §8.1, are good:

[4] This argument comes from §8.10 of Rindler (2006); Schutz has a more mathematical argument in his §8.1. Which you prefer is a matter of taste, but in keeping with our attempt to talk about physics in this chapter, we'll prefer the Rindler version for now.

> The element of controversy and doubt, that have continued to shroud the general theory of relativity to this day, derives precisely from this fact, namely that in the formulation of his theory Einstein incorporates aesthetic criteria; and every critic feels that he is entitled to his own differing aesthetic and philosophic criteria. Let me simply say that I do not share these doubts; and I shall leave it at that.

The one variation of Einstein's equation that is now being taken seriously is one that Einstein himself reluctantly suggested. Since $g_{\alpha\beta;\mu} = 0$ identically, we can add any constant multiple of the metric to the Einstein tensor without disturbing the right-hand side of Eq. (4.32). Specifically, we can write

$$G^{\mu\nu} + \Lambda g^{\mu\nu} = \kappa T^{\mu\nu}. \tag{4.33}$$

The extra term is referred to as the *cosmological constant*. Einstein introduced it in order to permit a static solution to the field equations, but the experimental evidence for the big bang showed that this was not in fact a requirement, and the parameter Λ was determined to be vanishingly small. Much more recently, however, studies of dark matter and the cosmic energy budget have shown that the large-scale structure of the universe is not completely determined by its matter content, baryonic or otherwise, and so Λ, in the form of 'dark energy', is now again the subject of detailed study.

The results of NASA's WMAP mission were the first, in 2003, to show that such a cosmological term, related to a dark energy field, is a necessary addition to Einstein's equations of Eq. (4.32) in order to match the universe we find ourselves in. Subsequent results have not changed this conclusion.

4.2.5 Degrees of Freedom

Einstein's equation[5] constitutes ten second-order nonlinear differential equations (ten since there are only ten independent components in the Einstein tensor), which reduce to six independent equations when we take account of the four differential identities of Eq. (4.22). Between them, these determine six of the ten independent components of the metric $g_{\mu\nu}$, with the remaining four degrees of freedom corresponding to changes in the four coordinate functions $x^{\mu}(P)$ (trivially, for example, I can decide to mark my coordinates in feet rather than metres, or rotate the coordinate frame, without changing the physics). The nonlinearity (meaning that adding together two solutions to the equation does *not* produce another solution) is what allows space-time to couple to

[5] As with the uncertain pluralisation of the Christoffel symbol(s), authors refer to Eq. (4.32) as both the Einstein equation and equations.

itself without the presence of any curvature terms in the energy-momentum tensor (which acts as the source of the field); it is also what makes Eq. (4.32) devilishly difficult to solve, and Appendix B is devoted to examining some of the solutions that have been derived over the years.

This, ultimately, is why we are so interested in the Ricci tensor, which otherwise seems like a bit of a mathematical curiosity in Section 4.2.1. It is the Ricci tensor, via the Einstein tensor, which is constrained by the (physically motivated) expression Eq. (4.32). The Riemann tensor shares these constraints, but has additional degrees of freedom that are of limited physical significance.

The identities Eq. (3.55) together reduce the number of independent components of the curvature tensor (with a metric connection) from 256 (4^4) to 20. That corresponds to there being 20 independent second derivatives of the metric $g_{\alpha\beta,\mu\nu}$ rather than 100 (100 since both the metric and partial differentiation are symmetric). See Exercise 4.5, and the end of Schutz §6.2.

[Exercise 4.5]

4.2.6 The Field Equations from a Variational Principle

The account earlier in the section, of how we obtain Einstein's equations, is pragmatic, and broadly follows Einstein's own approach to obtaining them. As well, it conveniently introduces the idea of the energy-momentum tensor, and lets us develop some intuitions about it. It is not the only way to obtain the equations, however.

In Section 3.4.1, we saw, in passing, how we could obtain the geodesic equation by extremising the integrated length of proper distance between two points, $\mathrm{d}s = |g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu|^{1/2}$. We can do something very similar with the *Einstein–Hilbert action*,

$$S = \frac{1}{16\pi}\int_\Omega R(-g)^{1/2}\mathrm{d}^4x. \tag{4.34}$$

Here, R is the Ricci curvature scalar of Eq. (4.17), g is the determinant of the metric, and the volume of integration, Ω is the region interior to some boundary where we can take the variation to be zero. The Ricci scalar is a simple object – a scalar field on space-time – that characterises the local curvature at each point. Under a change of basis, the volume element d^4x is scaled by a factor of the jacobian $|\partial x^{\bar{\mu}}/\partial x^\mu|$, and the determinant g by a factor of (jacobian)$^{-2}$ (these are 'tensor densities'), so that the quantity $\sqrt{-g}\,\mathrm{d}^4x$ is a scalar. Such volume elements, and the associated tensor densities, also appear in the analysis of the volume elements of Section 4.1.3.

We assume that this action, S, is *extremised* by the variation $\delta g^{\mu\nu}$ in the metric. This is a physical statement, and it is startling that such a simple statement – almost the simplest dimensionally consistent non-trivial statement we can make with these raw materials – combined with the very profound ideas of the calculus of variations, can lead us to the Einstein equations.

Calculating the variation, δS, resulting from a variation $\delta g^{\mu\nu}$, we find

$$\delta S = \int \mathrm{d}^4x\sqrt{-g}\left[R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu}\right]\delta g^{\mu\nu} \tag{4.35}$$

(the calculation is not long, but is somewhat tricky, and is described in Carroll [2004, §4.3], and in MTW [1973, box 17.2 and chapter 21]). You will recognise the term in square brackets from Eq. (4.23); requiring that $\delta S = 0$ for all variations $g_{\mu\nu}$ therefore implies that

$$G_{\mu\nu} = 0,$$

recovering Einstein's vacuum field equations.

We can add a second term S_M to the action, which depends on the energy-momentum content of the space-time volume, then perform the same calculation, and discover the field equations in the presence of matter. Choosing what that term S_M should be is of course an intricate matter, but if we obtain from it the tensor

$$T_{\mu\nu} = -2\frac{1}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}},$$

then we can recover the Einstein equations of Eq. (4.32).

4.3 The Newtonian Limit

We cannot finish this book without using at least one physical metric, and the one we shall briefly examine is the metric in the *weak field limit*, where space-time is curved only slightly, such as round a small object like the earth.

Before we do that we need to get units straight, and recap Section 1.4.1. In SR we chose our unit of time to be the metre, and we followed that convention in this book. That meant that the speed of light c was dimensionless and exact:

$$1 = c = 299\,792\,458\,\mathrm{m\,s^{-1}}.$$

In gravitational physics, we use natural units, for much the same reason. In SI units, Newton's gravitational constant has the dimensions $[G] = \mathrm{kg^{-1}m^3s^{-2}}$, but it is convenient in GR to have G dimensionless, and to this end we choose

our unit of *mass* to be the metre, with the conversion factor between this and the other mass unit, kg, obtained by:

$$1 = \frac{G}{c^2} = 7.425 \times 10^{-28}\,\mathrm{m\,kg^{-1}}.$$

See Schutz's §8.1 and Exercise 1.3 for a table of physical values in these units. Measuring masses in metres turns out unexpectedly intuitive: when you learn about (Schwarzschild) black holes you discover that the radius of the event horizon of an object is twice the value of the object's mass expressed in metres. Also, within the solar system, the mass of the sun is less precisely measurable than the value of the 'heliocentric gravitational constant', $GM_\odot$, which has units of $\mathrm{m^3\,s^{-2}}$ in SI units, and thus units of metres in natural units (the 'gravitational radius' of the sun $GM_\odot$ is known to one part in 10^{10}, but since G is known only to one part in 10^4 or so, the value of $M_\odot$ in kg has the same uncertainty).

In the weak-field approximation, we take the space-time round a small object to be nearly minkowskian, with

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}, \tag{4.36}$$

where $|h_{\alpha\beta}| \ll 1$, and the matrix $\eta_{\alpha\beta}$ is the matrix of components of the metric in Minkowski space. Note that Eq. (4.36), defining $h_{\alpha\beta}$, is a *matrix* equation, rather than a tensor one: we are choosing coordinates in which the matrix of components $g_{\alpha\beta}$ of the metric tensor g is approximately equal to $\eta_{\alpha\beta}$. If we Lorentz-transform Eq. (4.36) – using the $\Lambda^{\alpha}_{\bar{\alpha}}$ of SR, for which $\eta_{\bar{\alpha}\bar{\beta}} = \Lambda^{\alpha}_{\bar{\alpha}}\Lambda^{\beta}_{\bar{\beta}}\eta_{\alpha\beta}$ – we get an equation of the same form as Eq. (4.36), but in the new coordinates; that is, the components $h_{\alpha\beta}$ transform as if they were the components of a tensor in SR. This allows us to express $R^{\alpha}{}_{\beta\mu\nu}$, $R_{\alpha\beta}$ and $G^{\alpha\beta}$, and thus Einstein's equation itself, in terms of $h_{\alpha\beta}$ plus corrections of order $|h_{\alpha\beta}|^2$. The picture here is that $g_{\alpha\beta}$ is the result of a *perturbation* on flat (Minkowski) space-time, and that h (which encodes that perturbation) is a tensor in Minkowski space: expressing Einstein's equations in terms of h (accurate to first order in $h_{\alpha\beta}$) gives us a mathematically tractable problem to solve.

The next step is to observe that in the newtonian limit, which is the limit where Newton's gravity works, the gravitational potential $|\phi| \ll 1$ and speeds $|\mathbf{v}| \ll 1$. This implies that $|T^{00}| \gg |T^{0i}| \gg |T^{ij}|$ (because $T^{00} \propto m$, $T^{0i} \propto v^i$ and $T^{ij} \propto v^i v^j$, with $v_{\mathrm{earth}} \approx 10^{-4}$). We then identify $T^{00} = \rho + O(\rho v^2)$. By matching the resulting form of Einstein's equation with Newton's equation for gravity, we fix the constant κ in Eq. (4.32), so that, in geometrical units,

$$G^{\mu\nu} = 8\pi T^{\mu\nu}. \tag{4.37}$$

The solution to this equation, in this approximation, is

$$h^{00} = h^{11} = h^{22} = h^{33} = -2\phi, \tag{4.38}$$

which translates into a metric for newtonian space-time

$$\mathsf{g} \to \text{diag}(-(1+2\phi), 1-2\phi, 1-2\phi, 1-2\phi), \tag{4.39a}$$

which can be alternatively written as the interval

$$ds^2 = -(1+2\phi)dt^2 + (1-2\phi)(dx^2 + dy^2 + dz^2). \tag{4.39b}$$

See Schutz §§8.3–8.4 for the slightly intricate details of this derivation to Eq. (4.39), and see his §7.2 for the derivation of the newtonian geodesics of Section 4.3.2. Carroll (2004) gives an overlapping account of the same material in §4.1 and (very usefully, with more technical background) §7.1. We return to this approximation in Appendix B.

4.3.1 Why Is h Not a Tensor? A Digression on Gauge Symmetries

We said, in the discussion after Eq. (4.36), that $h_{\alpha\beta}$ is not a tensor, even though it looks like one, and in leading up to Eq. (4.39) we have treated it as one. Although Eq. (4.36) is a tensor equation (there does exist a tensor $g - \eta$) this is only a useful thing to do in a coordinate system in which $|h_{\alpha\beta} \ll 1|$. In that coordinate system, the approximations Eq. (4.38) and Eq. (4.39) are true to first order in $h_{\alpha\beta}$. That is, the components of $h_{\alpha\beta}$, as approximated, cannot be transformed into another coordinate system with an arbitrary transformation matrix Λ, and result in correct expressions.

There *is* a set of transformations that preserves the approximation, however, and it's useful to think a little more about this. Intuitively, a transformation to any other coordinate system in which $|h_{\bar{\alpha}\bar{\beta}} \ll 1|$ would produce an equivalent result. If we restrict Λ to the Lorentz transformations of SR, then the metric $\eta_{\alpha\beta}$ will be invariant, and the components $h_{\bar{\alpha}\bar{\beta}} = \Lambda^{\alpha}_{\bar{\alpha}}\Lambda^{\beta}_{\bar{\beta}}h_{\alpha\beta}$ will be small for small velocities: the approximation is still good in these new coordinates. We can think of $h_{\alpha\beta}$ as being a tensor within a background (Minkowski) space.

In slightly more formal terms – and see also Carroll (2004, section 7.1) – we can consider a vector field $\xi^{\mu}(x)$ on the background Minkowski space, and use this to generate a change from one coordinate system to another:

$$x^{\bar{\alpha}} = x^{\alpha} + \xi^{\alpha}(x), \tag{4.40}$$

giving

$$\frac{\partial x^{\alpha}}{\partial x^{\bar{\alpha}}} = \delta^{\alpha}_{\bar{\alpha}} - \xi^{\alpha}{}_{,\bar{\alpha}}.$$

Thus the metric in these new coordinates is

$$\begin{aligned} g_{\bar{\alpha}\bar{\beta}} &= \frac{\partial x^{\alpha}}{\partial x^{\bar{\alpha}}}\frac{\partial x^{\beta}}{\partial x^{\bar{\beta}}} g_{\alpha\beta} \\ &= (\delta^{\alpha}_{\bar{\alpha}} - \xi^{\alpha}{}_{,\bar{\alpha}})(\delta^{\beta}_{\bar{\beta}} - \xi^{\beta}{}_{,\bar{\beta}})(\eta_{\alpha\beta} + h_{\alpha\beta}). \end{aligned}$$

If we restrict the $\overline{\xi}$ to those for which $\xi^{\alpha}{}_{,\bar{\alpha}} \ll 1$, and retain only terms of leading order, then

$$g_{\bar{\alpha}\bar{\beta}} = \eta_{\bar{\alpha}\bar{\beta}} + (h_{\bar{\alpha}\bar{\beta}} - \xi_{\bar{\alpha},\bar{\beta}} - \xi_{\bar{\beta},\bar{\alpha}}). \tag{4.41}$$

Since $\xi^{\alpha}{}_{,\bar{\alpha}} \ll 1$, this has the same form as Eq. (4.36), meaning that vectors $\overline{\xi}$, which are 'small' in the sense discussed in this section, generate a family of coordinate systems, in all of which the metric is a perturbation ($|h_{\alpha\beta}| \ll 1$) on a background minkowskian space.[6]

You may also see this written using a '*symmetrisation*' notation,

$$A_{(ij)} \equiv (A_{ij} + A_{ji})/2, \tag{4.42}$$

which lets us write

$$g_{\bar{\alpha}\bar{\beta}} = \eta_{\bar{\alpha}\bar{\beta}} + h_{\bar{\alpha}\bar{\beta}} - 2\xi_{(\bar{\alpha},\bar{\beta})}.$$

There is a corresponding antisymmetrisation notation $A_{[ij]} \equiv (A_{ij} - A_{ji})/2$.

From Eq. (3.54), we promptly find

$$2R_{\alpha\beta\gamma\delta} = g_{\alpha\delta,\beta\gamma} - g_{\alpha\gamma,\beta\delta} + g_{\beta\gamma,\alpha\delta} - g_{\beta\delta,\alpha\gamma} \tag{4.43}$$

and (as you can fairly straightforwardly confirm) this does *not* change under the transformation $h_{\alpha\beta} \mapsto h_{\alpha\beta} - 2\xi_{(\alpha,\beta)}$.

This situation – in which we have identified a subspace of the general problem, in which the calculations are simpler, and physical quantities such as the Riemann tensor are invariant – is characteristic of a problem with a *gauge invariance*. If I describe and then solve a problem in classical newtonian mechanics, the dynamics of my solution will not change if I move the origin of my coordinates, or change units from metres to feet – that is, if I 're-gauge' the solution. A more mathematical way of putting this is that the Lagrangian is symmetric under the corresponding coordinate transformation, or that the

[6] The diffeomorphism in Eq. (4.41) is related to the Lie derivative mentioned in passing in section Section 3.3.2, which is in turn related to the idea of moving along the integral curves of the vector field $\xi^{\mu}(x)$.

degree of freedom that that transformation represents is not dynamically significant; and this gives me the freedom to select coordinates, from the continuous space of equivalent alternatives, in which the solution is easiest. You are trained to 'pick the right coordinates' from the earliest stages of your education in physics. In talking about the not-quite-tensor $h_{\alpha\beta}$ we are not picking a particular coordinate system, but instead implicitly identifying, in the set of metrics perturbatively different from the flat Minkowski metric, a set of equivalent coordinate systems. In particular, this set includes any coordinate system in which the nearly minkowskian space of interest is described, as in Eq. (4.36), by a minkowskian space plus small corrections.

You do the same thing when you discover that the predictions of Maxwell's equations are unchanged if you transform the magnetic vector potential $\mathbf{A}$ and electric potential ϕ by picking a function $\psi(\mathbf{x}, t)$ and changing $\mathbf{A} \mapsto \mathbf{A} + \nabla\psi$ and $\phi \mapsto \phi - \partial\psi/\partial t$. If, rather than using arbitrary ψ, we restrict ourselves to functions ψ that are such that $\nabla \cdot \mathbf{A} + \partial\phi/\partial t = 0$, then the remaining calculations become simpler; this is the Lorenz gauge condition.[7] This deliberate restriction of ourselves to only a subset of the possible functions $\mathbf{A}$ and ϕ, or of picking coordinates so that $|h_{\alpha\beta} \ll 1|$, is a more sophisticated version of 'picking the right coordinates', with the same motivation. We pick up this discussion of gauge conditions in Section B.3.

In summary, therefore: the decomposition of Eq. (4.36) can be viewed either in terms of tensors in a background (flat) space-time (as discussed earlier in this section), or as exploitation of a gauge freedom in GR. Because of the coordinate invariance of GR, we are free to choose coordinates (i.e., choose a gauge) in which the matrix $h_{\alpha\beta}$ has desirable (i.e., simplifying) properties. The details omitted here are to do with identifying what the desirable simplifications are, and proving that a suitable choice of coordinates is indeed always possible. The solution to Eq. (4.37) in terms of h can then be fairly directly shown to be Eq. (4.38).

4.3.2 Geodesics in Newtonian Space-time

What are the geodesics in this space-time? The geodesic equation is $\nabla_{\overline{U}}\overline{U} = 0$. This geodesic curve has affine parameter τ, but by rescaling this parameter though an affine transformation ($\tau \mapsto \tau/m$), we can express this in terms of

[7] Note the spelling: the Lorenz gauge is named after the Danish physicist Ludvig Lorenz, who is different from the Dutch physicist Hendrik Antoon Lorentz, after whom the (Poincaré–Larmor–FitzGerald–)Lorentz transformation is named. Your confusion about this is widely shared, possibly even by Lorentz (Nevels & Shin 2001). It doesn't help that there is also a Lorenz–Lorentz relation in optics, associated with both of them, in one order or another.

the momentum $\overline{p} = m\overline{U}$. This has the advantage that the resulting geodesic equation

$$\nabla_{\overline{p}}\overline{p} = 0 \tag{4.44}$$

is also valid for photons, which have a well-defined momentum even though they have no mass m. We shall now solve this equation, to find the path of a free-falling particle through this space-time.

The component form of Eq. (4.44) is

$$p^{\alpha}p^{\mu}{}_{;\alpha} = 0, \tag{4.45}$$

or

$$p^{\alpha}p^{\mu}{}_{,\alpha} + \Gamma^{\mu}_{\alpha\beta}p^{\alpha}p^{\beta} = 0. \tag{4.46}$$

If we restrict ourselves to the motion of a non-relativistic particle through this space-time, we have $|p^0| \gg |p^i|$, and we reduce this equation to

$$m\frac{\mathrm{d}}{\mathrm{d}\tau}p^{\mu} + \Gamma^{\mu}_{00}(p^0)^2 = 0. \tag{4.47}$$

The 0–0 Christoffel symbols for this metric, in this approximation, from Eq. (4.39), are

$$\Gamma^0_{00} = \phi_{,0} + O(\phi^2) \tag{4.48}$$

$$\Gamma^i_{00} = -\tfrac{1}{2}(-2\phi)_{,j}\delta^{ij}. \tag{4.49}$$

The 0-th component of Eq. (4.47) then tells us that

$$\frac{\mathrm{d}p^0}{\mathrm{d}\tau} = -m\frac{\partial\phi}{\partial\tau}, \tag{4.50}$$

so that the energy of the particle in this frame is conserved in a non-time-dependent field (the particle picks up kinetic energy as it falls, and loses gravitational potential energy). The space component is

$$\frac{\mathrm{d}p^i}{\mathrm{d}\tau} = -m\phi^{,i}, \tag{4.51}$$

which is simply Newton's law of gravitation, $\mathbf{f} = -m\nabla\phi$.

Thus we have come a long way in this book, from Special Relativity back, through Ptolemy and Newton, to well before the place we started. We have discovered that the universe is simple (the Equivalence Principle and Eq. (4.32)), and that we are now well placed to look upward and outward, towards the physical applications of General Relativity.

[Exercises 4.6 and 4.7]

I do not know what I may appear to the world, but to myself I seem to have been only like a boy playing on the seashore, and diverting myself in now and then finding a smoother pebble or a prettier shell than ordinary, whilst the great ocean of truth lay all undiscovered before me.

Isaac Newton, as quoted in Brewster, *Memoirs of the Life, Writings, and Discoveries of Sir Isaac Newton* .

Exercises

Exercise 4.1 (§ 4.1.2) Deduce Eq. (4.7), given that you have only the tensors $\overline{U} \otimes \overline{U}$ and $\mathsf{g} = \eta$ to work with, that the result must be proportional to both ρ and p, and that it must be consistent with both Eq. (4.6) and Eq. (4.4) in the limit $p = 0$. Thus write down the general expression $\mathsf{T} = (a\rho + bp)\overline{U} \otimes \overline{U} + (c\rho + dp)\mathsf{g}$ and apply the various constraints. Recall that $\overline{U} = (1, \mathbf{0})$ in the MCRF. $[u^+]$

Exercise 4.2 (§ 4.1.3) Calculate the components of $\widetilde{\sigma}$ and $\overline{A}$ after a Lorentz boost of speed v along $\mathbf{a}$. Recall that $A^{\bar{\mu}} = \Lambda^{\bar{\mu}}_{\mu} A^{\mu}$, where $\Lambda^{\bar{\mu}}_{\mu}$ is given by Eq. (2.34). Verify that $\langle \widetilde{\sigma}, \overline{A} \rangle = 0$ in this frame, too. $[d^-]$

Exercise 4.3 (§ 4.2.2) What would happen to an electric motor in free fall across the event horizon of a black hole (ignore any tidal effects)? $[d^-]$

Exercise 4.4 (§ 4.2.2) At various points in the development of the mathematical theory of GR, we pick a coordinate system in which differentiation is simple, and do a calculation using non-covariant differentiation, indicated by a comma. We then immediately deduce the covariant result, replacing this comma with a semicolon.

Recall that, the strong EP is sometimes referred to as the comma-goes-to-semicolon rule.

Without calculation, explain the logic of each of these replacements of a comma with a semicolon, putting particular stress on the distinction between them. $[u^+]$

Exercise 4.5 (§ 4.2.5) Prove that the curvature tensor has only 20 independent components for a 4-dimensional manifold, when you take equations Eqs. (3.55a) and (3.55b) into account.

Exercise 4.6 (§ 4.3.2) The geodesic equation, in terms of the momentum one-form $\widetilde{p}$, can be obtained by index-lowering Eq. (4.45) to obtain

$$p^{\alpha} p_{\beta;\alpha} = 0.$$

By expanding this, taking advantage of the symmetry of the resulting expression under index swaps, and using the relation $p^\alpha \mathrm{d}/\mathrm{d}x^\alpha = m\mathrm{d}/\mathrm{d}\tau$, show that

$$m\frac{\mathrm{d}p_\alpha}{\mathrm{d}\tau} = \tfrac{1}{2}g_{\beta\gamma,\alpha}p^\beta p^\gamma. \tag{i}$$

You may need the relations

$$p_{\alpha;\beta} = p_{\alpha,\beta} - \Gamma^\gamma_{\alpha\beta}p_\gamma$$
$$\Gamma^\gamma_{\alpha\beta} = \tfrac{1}{2}g^{\gamma\lambda}(g_{\lambda\alpha,\beta} + g_{\lambda\beta,\alpha} - g_{\alpha\beta,\lambda})$$

What does Eq. (i) tell you about geodesic motion in a non-time-varying metric? [u^+]

Exercise 4.7 (§ 4.3.2) The Schwarzschild metric is

$$g_{tt} = -(1-a) \qquad g_{rr} = (1-a)^{-1}$$
$$g_{\theta\theta} = r^2 \qquad g_{\phi\phi} = r^2\sin^2\theta,$$

where $a = 2M/r$, M is a constant, and all other metric components are zero.

Calculate the five non-zero derivatives of the metric.

Using Eq. (i), and by considering the relevant components of $\mathrm{d}p_\alpha/\mathrm{d}\tau$, demonstrate that:

1. if a particle is initially moving in the equatorial plane (that is, with $\theta = \pi/2$ and $p_\theta = 0$), then it remains in that plane;
2. if a particle is released from rest in these coordinates (that is, with $p_r = p_\theta = p_\phi = 0$, and $p_t \neq 0$), it initially moves radially inwards.

Appendix A Special Relativity – A Brief Introduction

I presume you have studied Special Relativity at some point. This appendix is intended to remind you of what you learned there, in a way that is notationally and conceptually homogeneous with the rest of the text here.

This appendix is intended to be standalone, but because it is necessarily rather compact, you might want to back it up with other reading. Taylor and Wheeler (1992) is an excellent account of Special Relativity (hereafter 'SR'), written in a style that is simultaneously conversational and rigourous (Wheeler, here, is the Wheeler of MTW (1973)). Rindler (2006) is, as mentioned elsewhere, now slightly old-fashioned in its treatment of GR, but is extremely thoughtful about the conceptual underpinnings of SR. In contrast, the first chapter of Landau and Lifschitz (1975) gives an admirably compact account of SR, which would be hard to learn from, but which could consolidate an understanding otherwise obtained.

There are several popular science books that are about, or that mention, relativity – these aren't to be despised just because you're now doing the subject 'properly'. These books tend to ignore any maths, and skip more pedantic detail (so they won't get you through an exam), but in exchange they spend their efforts on the underlying ideas. Those underlying ideas, and developing your intuition about relativity, are things that can sometimes be forgotten in more formal courses. I've always liked Schwartz and McGuinness (2003), which is a cartoon book but very clear (and I declare a sentimental attachment, since this is the book where I first learned about relativity); these books, like this appendix, and like many other introductions to relativity, partly follows Einstein's own popular account Einstein (1920).

A.1 The Basic Ideas

Relativity is simple. Essentially the only new physics that will be introduced here boils down to just:

1. All inertial reference frames are equivalent for the performance of all physical experiments (the Equivalence Principle);
2. The speed of light has the same constant value when measured in any inertial frame.

We must now (a) understand what these two postulates really mean and (b) examine both their direct consequences, and the way that we have to adjust the physics we already know.

A.1.1 Events

An 'event' in SR is something that happens at a particular place, at a particular instant of time. The standard examples of events are a flashbulb going off, or an explosion, or two things colliding.

Note that it is *events*, and not the reference frames that we are about to mention, that are primary. Events are real things that happen in the real world; the separations between events are also real; reference frames are a construct *we* add to events to allow us to give them numbers, and to allow us to manipulate and understand them. That is, events are not 'relative to an observer' or 'frame dependent' – everyone agrees that an event happens. SR is about how we reconcile the different measurements of an event, that different, relatively moving, observers make.

A.1.2 Inertial Reference Frames

A *reference frame* is simply a method of assigning a position, as a set of numbers, to events. Whenever you have a coordinate system, you have a reference frame. The coordinate systems that spring first to mind are possibly the (x, y, z) or (r, θ, ϕ) of physics problems. You can generate an indefinite number of reference frames, fixed to various things moving in various ways. However, we can pick out some frames as special, namely those frames that are not accelerating.

Imagine placing a ball at rest on a flat table: you'd expect it to stay in place; if you roll it across the table, it would move in a straight line. This is merely the expression of Newton's first law: 'bodies move in straight lines at constant velocity, unless acted on by an external force'. In what circumstances will this not be true?[1] If that table is on board a train that is accelerating out of a station, then the ball will start to roll towards the back of the train. This observation makes perfect sense from the point of view of someone on the station platform, who sees the ball as stationary, and the train being pulled from under it. The station is an inertial frame, and the accelerating train carriage is not.

This example illustrates what we will make more precise shortly, that position and speed are frame-dependent quantities, but acceleration is not. If you are sitting in a train carriage, then the force applied when it accelerates, which you might feel through the seat or measure using an 'accelerometer' such as a plumb line, is frame-independent.

In SR, inertial frames are infinite in extent; also, any pair of inertial frames are moving with a constant velocity with respect to each other. In GR, in contrast, inertial frames are necessarily *local*, in the sense of being meaningful only in the region

[1] By restricting ourselves to only horizontal motion, we evade any consideration of gravity. With that constraint, the definition of 'inertial frame' here is consistent with the broader definition appropriate to GR, which refers to a frame attached to a body in free fall, moving *only* under the influence of gravity.

surrounding a point of interest; and they may be 'accelerating' with respect to each other in the sense that the second derivative of position is non-zero, even though there is no acceleration measurable in the frame (think of two people in free fall on opposite sides of the earth).

[Exercise A.1]

A.1.3 Measuring Lengths and Times: Simultaneity

How do we measure times? Einstein put this as well as anyone else in 1905:

> We must take into account that all our judgments in which time plays a part are always judgments of *simultaneous events*. If, for instance, I say, 'That train arrives here at 7 o'clock,' I mean something like this: 'The pointing of the small hand of my watch to 7 and the arrival of the train are simultaneous events.' Einstein (1905)

In SR, all observations are of *events* that are adjacent to us. If two events happen at the same place and time – for example I set off a firecracker whilst looking at my wristwatch,[2] or two cars try to occupy the same location at the same time,[3] – then they are simultaneous for any observer who can see it: that metal was bent in a collision cannot possibly depend on who's looking at it or how they are moving when they see it. The space coordinate of the event is given by my (fixed and known) position within a frame. The time coordinate of an event is the time on my watch when I observe it, and my watch is synchronised with all the other clocks in the frame (one can go into a great deal of detail about this synchronisation process; see for example Rindler (2006, §2.5) and Taylor and Wheeler (1992, §2.6)).

If the event happens some distance away, however (answering a question such as 'what time does the train pass the next signal box?'), or if we want to know what time was measured by someone in a moving frame (answering, for example, 'what is the time on the train driver's watch as the train passes through the station?'), things are not so simple, as most of the rest of this text makes clear. We will typically imagine multiple observers at multiple events; indeed we imagine one local observer per frame of interest, stationary in that frame, and responsible for reporting the space and time coordinates of the event 'as measured in that frame'.

By making *only* local observations we avoid worrying about light-travel time. To make observations extended in space or time, we employ multiple observers. For example, we might measure the 'length of a rod' by subtracting the coordinates of two observers who were adjacent to opposite ends of the rod at the same prearranged time.

A.1.4 Standard Configuration

Finally, a bit of terminology to do with reference frames.

Two frames S and S', with spatial coordinates (x, y, z) and (x', y', z') and time coordinates t and t' are said to be in *standard configuration* (Figure A.1) if:

[2] An educational experience, in many ways, but one probably best kept as a thought experiment.
[3] Ditto.

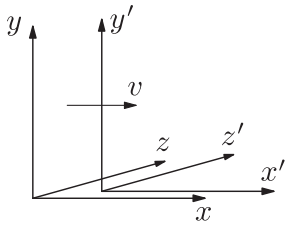

Figure A.1 Standard configuration.

1. they are aligned so that the (x, y, z) and (x', y', z') axes are parallel;
2. the frame S' is moving along the x axis with velocity V;
3. we set the zero of the time coordinates so that the origins coincide at $t = t' = 0$ (which means that the origin of the S' frame is always at position $x = Vt$).

When we refer to 'frame S' and 'frame S'', we will interchangably be referring either to the frames themselves, or to the sets of coordinates (t, x, y, z) or (t', x', y', z').

Frame S' will often be termed the *rest frame*; however, it should *always* be the rest frame *of something*. Yes, it does seem a little counterintuitive that it's the 'moving frame' that's the rest frame, but it's called the rest frame because it's the frame in which the thing we're interested in – be it a train carriage or an electron – is at rest. It's in the rest frame of the carriage that the carriage is measured to have its *rest length* or *proper length*.

A.2 The Postulates

Galileo described the Principle of Relativity in 1632, in *Dialogue Concerning the Two Chief World Systems*, using an elaborate image which invited the reader to imagine making observations of animals and falling water, and imagining jumping forward and aft, first in a ship in harbour, and then in the ship moving at a constant velocity, and finding oneself unable to tell the difference. This is a statement that 'you can't tell you're moving' – there is nothing you can do in the ship, or in a train or a plane, without looking outside, which will let you know whether you're stationary or moving at a constant velocity. More formally, and using the language in the previous section:

> *The Principle of Relativity:* All inertial frames are equivalent for the performance of *all* physical experiments.

That is, there is no place for the idea of a standard of absolute rest.

From the Relativity Principle (RP), one can show that, with certain obvious (but, as we shall discover, wrong) assumptions about the nature of space and time, one could derive the (apparently also rather obvious) *Galilean transformation* (GT)

$$x' = x - Vt \qquad y' = y \qquad z' = z \qquad t' = t \tag{A.1}$$

between two frames in the standard configuration of Section A.1.4. This transformation relates the coordinates of an event (t, x, y, z), measured in frame S, to the coordinates of the *same* event (t', x', y', z') in frame S'. Differentiating these, we find that

$$v'_x = v_x - V \qquad v'_y = v_y \qquad v'_z = v_z \qquad a' = a,$$

where v_x is the x-component of velocity, and so on.

If you take the RP as true, then it follows that any putative law of mechanics that *does* appear to allow you to distinguish between reference frames cannot in fact be a law of physics. That is, the RP, *in classical mechanics*, demands that all laws of mechanics be *covariant under the Galilean Transformation*. What that means is that physical laws take the same form whether they are expressed in the coordinates S or S', related by a GT.

Consider for example the constant-acceleration equation $x = v_0 t + at^2/2$. If we transform this into the moving frame using Eq. (A.1), we immediately find

$$x' = v'_0 t' + a' t'^2/2$$

– that is, we find exactly the same relation, as if we had simply put primes on each of the quantities. This is known as 'form invariance', or sometimes 'covariance', and indicates that (in this example) the expressions for x and x' have exactly the same *form*, with the only difference being that we have different numerical values for the coefficients and coordinates.

Maxwell's equations, however, are *not* invariant under a GT. The wave equation, and Maxwell's equations, do not transform into themselves under a GT, and take their simplest form (that is, their well-known form) only in a 'stationary' frame. Einstein noted that electrodynamics appeared to be concerned only with relative motion, and did not take a different form when viewed in a moving frame. His famous 1905 paper is very clear on this point ('On the Electrodynamics of Moving Bodies'), and the very first words of it are:

> It is known that Maxwell's electrodynamics – as usually understood at the present time – when applied to moving bodies, leads to asymmetries which do not appear to be inherent in the phenomena. Take, for example, the reciprocal electrodynamic action of a magnet and a conductor ... Einstein (1905)

This paper then briskly elevates the principle of relativity to the status of a postulate, and adds to it a second one, stating that the speed of light has the same value 'independent of the state of motion of the emitting body': no matter what sort of experiment you are doing, whether you are directly observing the travel time of a flash of light, or doing some interferometric experiment, the speed of light relative to your apparatus will always have the same numerical value. This is perfectly independent of how fast you are moving: it is independent of whichever inertial frame you are in, so that another observer, measuring the *same* flash of light from their moving laboratory, will measure the speed of light relative to *their* detectors to have exactly the same value.

There is no real way of *justifying* either of these postulates: it is simply a truth of our universe, and we can do nothing more than simply *demonstrate* its truth through experiment.

[Exercise A.2]

A.2.1 Further Details

It is fairly easy to discuss the transformation properties of the wave equation, slightly more involved for Maxwell's Equations. Bell (1987, chapter 9) discusses this, or rather the Lorentz transformation of Maxwell's Equations, in some depth. More advanced

textbooks on electromagnetic theory also tend to have sections on SR, which make this point more or less emphatically.

The aether drift experiments are discussed in most relativity textbooks. The `sci.physics.relativity` FAQ (Roberts and Schleif 2007) provides a large list of references to experimental corroboration of SR. For an interesting sociological and historical take on the Michelson-Morley experiments, and the context in which they were interpreted, see also Collins and Pinch, (1993, chapter 2). Barton (1999, §§3.1 & 3.4) presents the underlying ideas clearly and at length, discusses experimental corroboration, and provides ample further references.

The constancy of the speed of light is not the only second postulate you could have. You could take alternatives such as 'Maxwell's Equations are true', or 'Moving clocks run slow according to . . . ', or any other statement that picked out the phenomena of SR, and you could still derive the results of SR, including, for an encore, the constancy of the speed of light. However, this particular second postulate is a particularly simple and fundamental one, which is why it is much the best choice. Alternatively, you could choose as a second postulate something like 'c is infinite' or 'The Galilean Transformation is true', and derive from the pair of postulates the rest of the laws of classical mechanics. The point here is that each pair of postulates would give you a perfectly consistent theory – a perfectly possible world – but the galilean transformation is one that does not happen to match *our* world other than as a low-speed approximation.

Taking a more mathematical tack, Rindler (2006, §2.17), and Barton less abstractly (1999, §4.3), shows that the *only* linear transformations consistent with the Euclidicity and isotropy of inertial frames are the Galilean and Lorentz Transformations. A second postulate consisting of 'there is no upper limit to the speed of propagation of interactions' picks out the GT; the statement 'there is an upper speed limit' (which the first postulate implies is the same in all frames) instead picks out the Lorentz Transformation with a dependence on that constant speed, and saying '. . . and light moves at that speed' sets the value of the constant. See also Rindler's other remarks (2006, §2.7) on the properties of the Lorentz Transformation; Landau and Lifshitz (1975, §1) take this tack, and are as lucidly compact as ever.

Taking a more historical tack, I have quoted Einstein's own (translated) words, not because the argument depends on his authority (it doesn't), but firstly because he introduces the key arguments with admirable compactness, and secondly because it is a very rare example of the first introduction of a core physical theory still being intelligible after it has been absorbed into the bedrock of physics. [Exercise A.3]

A.3 Spacetime and the Lorentz Transformation

A.3.1 Length Contraction and Time Dilation

Imagine two observers with synchronised watches, standing at each end of a train carriage: Fred (at the front) and Barbara (at the back).[4] At a prearranged time '0', a

[4] This argument ultimately originates from Einstein's popular book about relativity (Einstein, 1920), first published in English in 1920. It reappears in multiple variants, in planes, trains, automobiles, and rockets, in many popular and professional accounts of relativity. The variant described here is most directly descended from Rindler's version (2006).

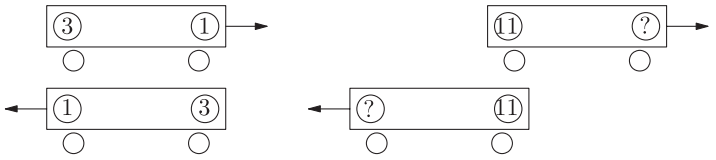

Figure A.2 Passing trains: (a, left) flash reaches rear of carriage; (b, right) rear observers coincide.

flashbulb fires at the centre of the carriage and the observers record the time the flash reaches them. Since Fred and Barbara are equidistant from the bulb, their times must be the same, for example time '3' units. In other words, Fred's and Barbara's watches both reading '3', are *simultaneous* events in the frame of the carriage.

Observing from the platform, we would see the light from the flash move both forward towards Fred and backwards towards Barbara, but at the same speed c, as measured on the platform. Consequently, the flash would naturally get to Barbara first. If, standing on the platform, you were to take a photograph at this point, you would get something like the upper part of Figure A.2a. Barbara's watch *must* read '3', since the flash meeting her and her watch reading '3' are simultaneous at the same point in space, and so must be simultaneous for observers in any frame. But at this point, the light moving towards Fred cannot yet have caught up with him: since the light reaches Fred when *his* watch reads '3', his watch must still be reading something less than that, '1', say. In other words, Barbara's watch reading '3' and Fred's watch reading '1' are simultaneous events in the inertial frame of the platform.

Now imagine observing two such trains go past, timetabled such that we can obtain the observations in Figure A.2a, where the light has reached both rear observers and neither front one. Now pause a moment, and take another photograph when the two rear observers are beside each other, this time getting Figure A.2b.

Barbara can report observing the front of the other carriage passing at time '3', whereas Fred reports the back of that carriage passing earlier, at time '1'. They can therefore conclude that they have measured the length of the other carriage and found it to be shorter than their own one. This is *length contraction*.

Similarly, Fred observes the rear clock in Figure A.2a as being two units fast, compared to his own. But Barbara can later, in Figure A.2b, observe that same clock to be reading '11' at the same time as hers, no longer fast. They know their own clocks were synchronised, so they can conclude that the rear clock in the other carriage was going more slowly than their clocks. This is *time dilation*.

Notably, Barbara and Fred's counterparts in the other carriage would come to precisely the same conclusions. Because this setup is perfectly symmetrical, they would measure Barbara and Fred's clocks to be moving slowly, and their carriage to be shorter. There is no sense in which one of the carriages is *absolutely* shorter than the other.

A.3.2 The Light Clock

We can put numbers to the effects.

The light clock (see Figure A.3) is an idealised timekeeper, in which a flash of light leaves a bulb, bounces off a mirror, and returns – this is one 'tick' of the

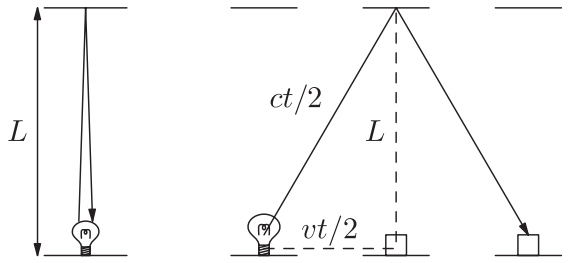

Figure A.3 The light clock, shown in its rest frame (left) and as observed in a frame in which the clock is moving at speed v (right).

clock. If the mirror and the flashbulb are a distance L apart, then $2L = ct'$, where t' is the time on the watch of an observer standing by, and moving with, the clock, and c is the frame-independent speed of light. Also note that the clock's mirrors are arranged perpendicular to the clock's motion, and both the stationary and the moving observer measure the same separation between them – there is no length contraction perpendicular to the motion.[5] Examining the same tick in a frame in which the clock is moving, we find $(ct/2)^2 = L^2 + (vt/2)^2$ and thus, using the expression for L above,

$$t' = \frac{t}{\gamma}, \tag{A.2}$$

where the factor $\gamma = \gamma(v)$ is defined as

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}. \tag{A.3}$$

Now, the important thing about this equation is that it involves t', the time for the clock to 'tick' as measured by the person standing next to it on the train, and it involves t, the time as measured by the person on the platform, and *they are not the same*

A.3.3 The Invariant Interval

If we take the light flash (call it event 1) to have coordinates $t'_1 = x'_1 = 0$ in the clock's frame, and coordinates $t_1 = x_1 = 0$ in the 'lab' frame (Section A.1.4), then the detection of the reflection (event 2) has coordinates $x'_2 = 0$, $t'_2 = 2L/c$, $x_2 = vt_2$ and $t_2 = \gamma t'_2 = 2\gamma L/c$. If we now calculate $\Delta x'^2 - c^2\Delta t'^2$ or $\Delta x^2 - c^2\Delta t^2$ we obtain $(2L)^2$ in both cases. This is not a coincidence.

In general, if we calculate the corresponding separation between any two events, then we will obtain the same value irrespective of which frame's coordinates we use. This quantity is *frame invariant*. We have illustrated this here rather than proved it, but there is a proof, which depends only on the two postulates of SR, in Schutz (2009, §1.6)

[5] If there *were* a perpendicular length contraction, then observers would see a passing relativistic train's axles contract so that they derailed inside the tracks; however observers on the train would see the passing sleepers contract, so that the train would derail outside the tracks; the contradiction implies there can be no such contraction.

or in Landau and Lifshitz (1975). Alternatively, assuming this in place of the second postulate would allow us to deduce the constancy of the speed of light.

This quantity $\Delta s^2 = \Delta x^2 - c^2\Delta t^2$ is referred to as the *interval*, or sometimes, interchangeably, as the *squared interval* or the *invariant interval*.

From here on, we will handle coordinates only in *natural units*, in which $c = 1$, with the result that we define

$$\Delta s^2 = \Delta x^2 - \Delta t^2. \tag{A.4}$$

Some authors define the interval with the opposite sign. The definition here is compatible with Schutz but opposite to Rindler.

A.3.4 The Lorentz Transformation

We have now obtained results describing length contraction and time dilation. We now wish to find a way of relating the coordinates of any event, as measured in any pair of frames in relative motion. That relation – a transformation from one coordinate system to another – is the *Lorentz Transformation* (LT).

Specifically, consider two frames in standard configuration, and imagine an event such as a flashbulb going off; observers in the frames S and S' will give this event different coordinates (t, x, y, z), and (t', x', y', z'). How are these coordinates related?

First, we can note that $y' = y$ and $z' = z$ (this is just a restatement of the lack of a perpendicular length contraction, discussed in Section A.3.2). Therefore we can take the event to happen on the x-axis, at $y = z = 0$.

Now imagine a second event, located at the origin with coordinates $(t, x) = (0, 0)$ in frame S and $(t', x') = (0, 0)$ in frame S'. Since we have two events, we have an interval between them, with the value $s^2 = (x - 0)^2 - (t - 0)^2 = x^2 - t^2$ in frame S. Since the interval is frame-independent, the calculation of this interval done by the observer in the primed frame will produce the same value:

$$x^2 - t^2 = s^2 = x'^2 - t'^2. \tag{A.5}$$

Thus the relationship between (t, x) and (t', x') must be one for which Eq. (A.5) is true.

If we consider two frames on the xy plane related by a rotation, then their coordinates will be related by

$$\begin{aligned} x' &= x\cos\theta + y\sin\theta \\ y' &= -x\sin\theta + y\cos\theta. \end{aligned} \tag{A.6}$$

and this will preserve the euclidean distance $r^2 = x^2 + y^2$. This is strongly reminiscent of Eq. (A.5), and we can make it more so by writing $l = \mathrm{i}t$ and $l' = \mathrm{i}t'$, so that Eq. (A.5) becomes

$$x^2 + l^2 = s^2 = x'^2 + l'^2. \tag{A.7}$$

This strongly suggests that the pairs (l, x) and (l', x') can be related by writing down the analogue of Eq. (A.6), replacing $y \mapsto l$ and $y' \mapsto l'$, for some angle θ that depends on v, the relative speed of frame S' in frame S. That is, this specifies a linear relation for which Eq. (A.5) is true. If we finally write $\theta = \mathrm{i}\phi$ (since l is pure imaginary, so is θ, so that ϕ

is real), and recall the trigonometric identities $\sin \mathrm{i}\phi = \mathrm{i} \sinh \phi$ and $\cos \mathrm{i}\phi = \cosh \phi$, then this expression for x' and l' becomes

$$x' = x \cosh \phi - t \sinh \phi \tag{A.8a}$$

$$t' = -x \sinh \phi + t \cosh \phi. \tag{A.8b}$$

Now consider an event at $x' = 0$ for some unknown t'. This happens at time t in the unprimed frame and thus happens at position $x = vt$ in that frame, in which case Eq. (A.8a) can be rewritten as

$$\tanh \phi (v) = v. \tag{A.9}$$

Since we now have ϕ as a function of v, we have, in Eq. (A.8), the full transformation between the two frames; combining these with a little hyperbolic trigonometry (remember $\cosh^2 \phi - \sinh^2 \phi = 1$), we can rewrite Eq. (A.8) in the more usual form (for $c = 1$)

$$t' = \gamma (t - vx) \tag{A.10a}$$

$$x' = \gamma (x - vt), \tag{A.10b}$$

where the trivial transformations for y and z complete the LT, and (as in Eq. (A.3) but now with $c = 1$)

$$\gamma = \left(1 - v^2\right)^{-1/2}. \tag{A.11}$$

If frame S' is moving with speed v relative to S, then S must have a speed $-v$ relative to S'. Swapping the roles of the primed and unprimed frames, the transformation from frame S' to frame S is exactly the same as Eq. (A.10), but with the opposite sign for v:

$$t = \gamma (t' + vx') \tag{A.12a}$$

$$x = \gamma (x' + vt'), \tag{A.12b}$$

which can be verified by direct solution of Eq. (A.10) for the unprimed coordinates.

A more direct, but less physically illuminating, route to the LT is to note that the transformation from the unprimed to the primed coordinates must be linear, if the equations of physics are to be invariant under a shift of origin. That is, we must have a transformation like $t' = Ax + By + Cz + Dt$, and similarly (with different coefficients) for x', y' and z'. By using RP and the constancy of the speed of light, one can deduce the transformation given in Eq. (A.10). See Rindler (2006, §2.6), Taylor and Wheeler (1992, §§L.4–L.5), or Barton (1999, §4.3) for details. Rindler (2006, §2.17) shows an even more powerful consequence of the same ideas. If we use this argument to deduce that the relationship must be $x' = \gamma (x - vt)$, for some unknown constant γ (Rindler §2.6), then we can instead deduce the LT, and the expression for γ, from the construction in Section A.3.3 by considering an event ④ located at the front of the carriage at time $t_4 = 0$.

A.3.5 Addition of Velocities

Adding and subtracting the expressions in Eq. (A.8), and recalling that $e^{\pm\phi} = \cosh\phi \pm \sinh\phi$, we find

$$t' - x' = e^{\phi}(t - x) \tag{A.13a}$$

$$t' + x' = e^{-\phi}(t + x), \tag{A.13b}$$

as yet another form (once we add $y' = y$ and $z' = z$) of the LT.

If we now have three frames, S, S' and S'' where the mutual velocities of S' vs. S, and S'' vs. S', are v_1 and v_2 respectively, how does the velocity v of S'' in S depend on v_1 and v_2 (being in standard configuration implies that $\mathbf{v}_1$ is parallel to $\mathbf{v}_2$)? Applying Eq. (A.13a) twice produces

$$t'' - x'' = e^{\phi}(t - x) = e^{\phi_1+\phi_2}(t - x), \tag{A.14}$$

where ϕ, ϕ_1, and ϕ_2 are the hyperbolic velocity parameters corresponding to v, v_1 and v_2. This shows us how to add velocities: Eq. (A.9) plus a little more hyperbolic trigonometry ($\tanh(\phi_1 + \phi_2) = (\tanh\phi_1 + \tanh\phi_2)/(1 + \tanh\phi_1 \tanh\phi_2)$) produces

$$v = \frac{v_1 + v_2}{1 + v_1 v_2}. \tag{A.15}$$

The form of the LT shown in Eq. (A.13), and the addition law in Eq. (A.14), conveniently indicate three interesting things about the LT: (i) for any two transformations performed one after the other, there exists a third with the same net effect (i.e., the LT is 'transitive'); (ii) there exists a transformation (with $\phi = 0$) that maps (t,x) to themselves (i.e., there exists an identity transformation); (iii) for every transformation (with $\phi = \phi_1$, say) there exists another transformation (with $\phi = -\phi_1$) that results in the identity transformation (i.e., there exists an inverse). These three properties are enough to indicate that the LT is an example of a mathematical 'group', known as the *Lorentz group*.

Any two IFs, not just those in standard configuration, may be related via a sequence of transformations, namely a translation to move the origin, a rotation to align the axes along the direction of motion, a LT, another rotation, and another translation. The transformation that augments the LT with translations and rotations is known as a *Poincaré transformation*, and it is a member of the *Poincaré group*.

A.3.6 Proper Time and the Invariant Interval

In some frame S, consider two events, one at the origin, and one on the x-axis at coordinates (t,x); take the two events to be timelike-separated, so that it is possible for some clock, moving at a constant velocity $v < c$, to be present at both events. In a frame attached to the clock, the spatial separation of the two events is zero, and the time coordinate in this frame is the time shown on the clock, which we shall label τ and call the *proper time*. Given the coordinates (t,x) in S, we can use Eq. (A.10a) to transform to the clock's rest frame and write

$$\tau = \gamma(t - vx). \tag{A.16}$$

This calculated proper time would be agreed on by all the observers who could not agree on the spatial or temporal separations of the two events. In other words, this number τ is invariant under a LT – it is a *Lorentz scalar*.

In the clock's frame, the interval separating these two events is just $s^2 = -\tau^2$, so that the invariance of the proper time is just another manifestation of the invariance of the interval, Eq. (A.4).

A.4 Vectors, Kinematics, and Dynamics

Section A.3 was concerned with static events as observed from moving frames. In this part, we are concerned with particle motion.

A.4.1 Kinematics

Consider a prototype displacement vector $(\Delta x, \Delta y, \Delta z)$. These are the components of a vector with respect to the usual axes $\mathbf{e}_x$, $\mathbf{e}_y$, and $\mathbf{e}_z$. We can rotate these into new axes $\mathbf{e}'_x$, $\mathbf{e}'_y$, and $\mathbf{e}'_z$ using one or more rotation matrices, and obtain coordinates $(\Delta x', \Delta y', \Delta z')$ for the same displacement vector, with respect to the new axes. These primed coordinates are different from the unprimed ones, but systematically related to them.

In four-dimensional spacetime, the prototype displacement 4-vector is $\Delta\overline{R} = (\Delta t, \Delta x, \Delta y, \Delta z)$, relative to the space axes and wristwatch of a specific observer, and the transformation that takes one 4-vector into another is the familiar LT of Eq. (A.10), or

$$\begin{pmatrix} \Delta t' \\ \Delta x' \\ \Delta y' \\ \Delta z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \Delta t \\ \Delta x \\ \Delta y \\ \Delta z \end{pmatrix} \tag{A.17}$$

with an inverse transformation obtained by swapping $v \leftrightarrow -v$. These give the coordinates of the same displacement as viewed by a second observer whose frame is in standard configuration with respect to the first.

We recognise as a 4-vector any geometrical object, the components of which, namely (A^0, A^1, A^2, A^3), transform in the same way as the coordinate transformation of Eq. (A.17).

It is no more than fiddly to extend Eq. (A.17) to Lorentz boosts that are not along the x-axis.

A 3-rotation, when applied to a 3-vector $\mathbf{A}$, conserves the euclidean length-squared of $\mathbf{A}$, $(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$, or, more generally, conserves the value of the inner product of two 3-vectors, $\mathbf{A} \cdot \mathbf{B}$. In exactly the same way, the transformation Eq. (A.17) conserves the corresponding inner product for 4-vectors,

$$\overline{A} \cdot \overline{B} = -A^0 B^0 + A^1 B^1 + A^2 B^2 + A^3 B^3 \tag{A.18a}$$

$$= \sum_{\mu,\nu} \eta_{\mu\nu} A^\mu B^\nu, \tag{A.18b}$$

where the matrix $\eta_{\mu\nu}$ is defined as

$$\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1). \tag{A.19}$$

The inner product of a vector with itself, $\overline{A} \cdot \overline{A}$, is its *norm* (or length-squared), and from this definition we can see that the norm of the displacement vector is $\Delta\overline{R} \cdot \Delta\overline{R} = -\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2 = \Delta s^2$, compatible with Eq. (A.5).

If the norm is positive, negative, or zero, then the vector (or the displacement) is termed respectively spacelike, timelike, or null. Two vectors are *orthogonal* if their inner product vanishes; it follows that in this geometry, a null vector (with $\overline{A} \cdot \overline{A} = 0$) is orthogonal to itself.

This matrix η is the *metric* of SR. Just as with the invariant interval (which is really just a consequence of this definition), some authors define the metric with the opposite signs.

A.4.2 Velocity and Acceleration

Since the displacement 4-vector $\Delta\overline{R}$ is a vector (in the sense that it transforms properly according to Eq. (A.17)), so is the infinitesimal displacement $\mathrm{d}\overline{R}$; since the proper time τ (see Section A.3.6) is a Lorentz scalar, we can divide each component of this infinitesimal displacement by the proper time and still have a vector. This latter vector is the *4-velocity*:

$$\overline{U} = \frac{\mathrm{d}\overline{R}}{\mathrm{d}\tau} = \left(\frac{\mathrm{d}x^0}{\mathrm{d}\tau}, \frac{\mathrm{d}x^1}{\mathrm{d}\tau}, \frac{\mathrm{d}x^2}{\mathrm{d}\tau}, \frac{\mathrm{d}x^3}{\mathrm{d}\tau}\right), \tag{A.20}$$

where we have written the components of $\mathrm{d}\overline{R}$ as $\mathrm{d}x^\mu$. We can write this as $U^\mu = \mathrm{d}x^\mu/\mathrm{d}\tau$. By the same argument, the 4-acceleration $A^\mu = \mathrm{d}U^\mu/\mathrm{d}\tau = \mathrm{d}^2x^\mu/\mathrm{d}\tau^2$ is a 4-vector, also.

Let us examine these components in more detail. The infinitesimal proper time is $\mathrm{d}\tau^2 = \mathrm{d}t^2 - |\mathrm{d}\mathbf{r}|^2$, so that

$$\left(\frac{\mathrm{d}\tau}{\mathrm{d}t}\right)^2 = \frac{(\mathrm{d}\tau)^2}{(\mathrm{d}t)^2} = 1 - \frac{|\mathrm{d}\mathbf{r}|^2}{(\mathrm{d}t)^2} = 1 - v^2 = \frac{1}{\gamma^2}.$$

From this, promptly,

$$U^0 = \frac{\mathrm{d}x^0}{\mathrm{d}\tau} = \frac{\mathrm{d}t}{\mathrm{d}\tau} = \gamma \tag{A.21a}$$

$$U^i = \frac{\mathrm{d}x^i}{\mathrm{d}\tau} = \frac{\mathrm{d}t}{\mathrm{d}\tau}\frac{\mathrm{d}x^i}{\mathrm{d}t} = \gamma v^i. \tag{A.21b}$$

You can view Eq. (A.21a) as yet another manifestation of time dilation. Thus we can write

$$\overline{U} = (\gamma, \gamma v_x, \gamma v_y, \gamma v_z) = \gamma\,(1, v_x, v_y, v_z) = \gamma\,(1, \mathbf{v}), \tag{A.22}$$

using $\mathbf{v}$ to represent the three (space) components of the (spatial) velocity vector, and where the last expression introduces a convenient notation.

In a frame that is co-moving with a particle, the particle's velocity is $\overline{U} = (1, 0, 0, 0)$, so that, from Eq. (A.18), $\overline{U} \cdot \overline{U} = -1$; since the inner product is frame invariant, it must have this same value in all frames, so that, quite generally, we have the relation

$$\overline{U} \cdot \overline{U} = -1. \tag{A.23}$$

You can confirm that this is indeed true by applying Eq. (A.18) to Eq. (A.22).

Here, we defined the 4-velocity by differentiating the displacement 4-vector, and deduced its value in a frame co-moving with a particle. We can now turn this on its head, and *define* the 4-velocity as a vector that has norm – 1 and that points along the *t*-axis of a co-moving frame (this is known as a 'tangent vector', and is effectively a vector 'pointing along' the world line). We have thus defined the 4-velocity of a particle as the vector that has components $(1, \mathbf{0})$ in the particle's rest frame. Note that the norm of the vector is always the same; the particle's speed relative to a frame S is indicated not by the 'length' of the velocity vector – its norm – but by its direction in S. We can then *deduce* the form in Eq. (A.22) as the Lorentz-transformed version of $(1, \mathbf{0})$.

Equations A.21 can lead us to some intuition about what the velocity vector is telling us. When we say that the velocity vector in the particle's rest frame is $(1, \mathbf{0})$, we are saying that, for each unit proper time τ, the particle moves the same amount through coordinate time t, and not at all through space x; the particle 'moves into the future' directly along the t-axis. When we are talking instead about a particle that is moving with respect to some frame, the equation $U^0 = \mathrm{d}t/\mathrm{d}\tau = \gamma$ tells us that the particle moves through a greater amount of this frame's coordinate time t, per unit proper time (where, again, the 'proper time' is the time showing on a clock attached to the particle). This is another appearance of 'time dilation'.

By further differentiating the components of the velocity $\overline{U}$, we can obtain the components of the acceleration 4-vector

$$\overline{A} = \gamma\,(\dot{\gamma}, \gamma\mathbf{v} + \gamma\mathbf{a})\,. \tag{A.24}$$

This is useful less often than the velocity vector, but it is fairly straightforward to deduce that $\overline{U} \cdot \overline{A} = 0$, and that $\overline{A} \cdot \overline{A} = a^2$, defining the *proper acceleration* a as the magnitude of the acceleration in the instantaneously co-moving inertial frame.

Finally, given two particles with velocities $\overline{U}$ and $\overline{V}$, and given that the second has velocity v with respect to the first, then in the first particle's rest frame the velocity vectors have components $\overline{U} = (1, \mathbf{0})$ and $\overline{V} = \gamma(v)(1, \mathbf{v})$. Thus

$$\overline{U} \cdot \overline{V} = \gamma(v),$$

and this inner product is, again, frame-independent.

A.4.3 Dynamics: Energy and Momentum

In the previous section, we have learned how to describe motion; we now want to explain it. In newtonian mechanics, we do this by defining quantities such as momentum, energy, force, and so on. We wish to find the Minkowski-space analogues of these.

We can start with momentum. We know that in newtonian mechanics, momentum is defined as mass times velocity. We have a velocity, so we can try defining a momentum 4-vector as

$$\overline{P} = m\overline{U} = m\gamma\,(1, \mathbf{v}). \tag{A.25}$$

Since m is a scalar, and $\overline{U}$ is a 4-vector, $\overline{P}$ must be a 4-vector also. Remember also that γ is a function of $\mathbf{v}$: $\gamma(\mathbf{v})$.

In the rest frame of the particle, this becomes $\overline{P} = m(1, \mathbf{0})$: it is a 4-vector whose norm ($\overline{P} \cdot \overline{P}$) is $-m^2$, and which points along the particle's world line. That is, it points in the direction of the particle's movement in space time. Since this is a vector, its norm and its direction are frame-independent quantities, so a particle's 4-momentum vector *always points in the direction of the particle's world line*, and the 4-momentum vector's norm is always $-m^2$. We'll call this vector the *momentum (4-)vector*, but it's also called the *energy-momentum vector*, and Taylor and Wheeler (1992) call it the *momenergy vector* (coining the word in an excellent chapter on it) in order to stress that it is not the same thing as the energy or momentum (or mass) that you are used to.

Note that here, and throughout, the symbol m denotes the mass as measured in a particle's rest frame. The reason I mention this is that some treatments of relativity, particularly older ones, introduce the concept of the 'relativistic mass', distinct from the 'rest mass'. The only (dubious) benefit of this is that it makes a factor of γ disappear from a few equations, making them look at little more like their newtonian counterparts; the cost is that of introducing one more new concept to worry about, which doesn't help much in the long term, and which can obscure aspects of the energy-momentum vector. Rindler introduces the relativistic mass; Taylor and Wheeler and Schutz don't.

Now consider a pair of incoming particles $\overline{P}_1$ and $\overline{P}_2$, which collide and produce a set of outgoing particles $\overline{P}_3$ and $\overline{P}_4$. Suppose that the total momentum is conserved:

$$\overline{P}_1 + \overline{P}_2 = \overline{P}_3 + \overline{P}_4. \tag{A.26a}$$

This is an equation between 4-vectors. Equating the time and space coordinates separately, recalling Eq. (A.25), and writing $\mathbf{p} \equiv \gamma m \mathbf{v}$, we have

$$m_1\gamma(v_1) + m_2\gamma(v_2) = m_3\gamma(v_3) + m_4\gamma(v_4) \tag{A.26b}$$

$$\mathbf{p}_1 + \mathbf{p}_2 = \mathbf{p}_3 + \mathbf{p}_4. \tag{A.26c}$$

Now recall that, as $v \to 0$, we have $\gamma(v) \to 1$, so that, from Eq. (A.25), the low-speed limit of the spatial part of the vector $\overline{P}$ is just $m\mathbf{v}$, so that the spatial part of the conservation equation, Eq. (A.26c), reduces to the statement that $m\mathbf{v}$ is conserved. Both of these prompt us to identify the spatial part of the 4-vector $\overline{P}$ as the familiar linear 3-momentum, and to justify giving $\overline{P}$ the name 4-momentum.

What, then, of the time component of Eq. (A.25)? Let us (with, admittedly, a little foreknowledge) write this as $P^0 = E$, so that

$$E = \gamma m. \tag{A.27}$$

If we now expand γ into a Taylor series, then

$$E = m + \tfrac{1}{2}mv^2 + O(v^4). \tag{A.28}$$

Now $\frac{1}{2}mv^2$ is the expression for the kinetic energy in newtonian mechanics, and Eq. (A.26b), compared with Eq. (A.27), tells us that this quantity E is conserved in collisions, so we have persuasive support for identifying the quantity E in Eq. (A.27)

as the relativistic *energy* of a particle with mass m and velocity v. If, finally, we rewrite Eq. (A.27) in physical units, we find

$$E = \gamma mc^2, \tag{A.29}$$

the low-speed limit of which (remember $\gamma(0) = 1$) recovers what has been called the most famous equation of the twentieth century.

The argument presented here after Eq. (A.26) has been concerned with giving names to quantities, and, reassuringly for us, linking those newly named things with quantities we already know about from newtonian mechanics. This may seem arbitrary, and it is certainly not any sort of proof that the 4-momentum is conserved as Eq. (A.26) says it might be. No proof is necessary, however: it turns out *from experiment* that Eq. (A.26) is a law of nature, so that we could simply include it as a postulate of relativistic dynamics and proceed to use it without bothering to identify its components with anything we are familiar with.

In case you are worried that we are pulling some sort of fast one, that we never had to do in newtonian mechanics, note that we *do* have to do a similar thing in newtonian mechanics. There, we postulate Newton's third law (action equals reaction), and from this we can deduce the conservation of momentum; here, we postulate the conservation of 4-momentum, and this would allow us to deduce a relativistic analogy of Newton's third law (I don't discuss relativistic force here, but it is easy to define). The postulational burden is the same in both cases.

We can see from Eq. (A.27) that, even when a particle is stationary and $v = 0$, the energy E is non-zero. In other words, a particle of mass m has an energy γm associated with it simply by virtue of its mass. The low-speed limit of Eq. (A.26b) simply expresses the conservation of mass, but we see from Eq. (A.27) that it is actually expressing the conservation of energy. In SR there is no real distinction between mass and energy – mass is, like kinetic, thermal, and strain energy, merely another form into which energy can be transmuted – albeit a particularly dense store of energy, as can be seen by calculating the energy equivalent, in Joules, of a mass of one kilogramme. It turns out from GR that it is not mass that gravitates, but energy-momentum (most typically, however, in the particularly dense form of mass), so that thermal and electromagnetic energy, for example, and even the energy in the gravitational field itself, all gravitate. (It is the non-linearity implicit in the last remark that is part of the explanation for the mathematical difficulty of GR.)

Let us now consider the norm of the 4-momentum vector. Like any such norm, it will be frame invariant, and so will express something fundamental about the vector, analogous to its length. Since this is the momentum vector we are talking about, this norm will be some important invariant of the motion, indicating something like the 'quantity of motion'. From the definition of the momentum, Eq. (A.25), and its norm, Eq. (A.23), we have

$$\overline{P} \cdot \overline{P} = m^2 \overline{U} \cdot \overline{U} = -m^2, \tag{A.30}$$

and we find that this important invariant is the mass of the moving particle.

Now using the definition of energy, Eq. (A.27), we can write $\overline{P} = (E, \mathbf{p})$, and find

$$\overline{P} \cdot \overline{P} = -E^2 + \mathbf{p} \cdot \mathbf{p}. \tag{A.31}$$

Writing now $p^2 = \mathbf{p} \cdot \mathbf{p}$, we can combine these to find

$$m^2 = E^2 - p^2. \tag{A.32}$$

This is not simply a handy way of relating E, p, and m. The 4-momentum $\overline{P}$ encapsulates the important features of the motion in the energy and spatial momentum. Though the latter are frame-dependent separately, they combine into a frame-independent quantity.

It seems odd to think of a particle's momentum as being always non-zero, irrespective of how rapidly it's moving; this is the same oddness that has the particle's velocity always being of length 1. One way of thinking about this is that it shows that the 4-momemtum vector (or energy-momentum or momenergy vector) is a 'better' thing than the ordinary 3-momentum: it's frame-independent, and so has a better claim to being something intrinsic to the particle. Another way of thinking about it is to conceive of the 4-velocity as showing the particle's movement through spacetime. In a frame in which the particle is 'at rest', the particle is in fact moving at speed c into the future. If you are looking at this particle from another frame, you'll see the particle move in space (it has non-zero space components to its 4-velocity in your frame), and it will consequently (in order that $\overline{U} \cdot \overline{U} = -1$) have a *larger* time component in your frame than it has in its rest frame. In other words, the particle moves through more time in your frame than it does in its rest frame – another manifestation of time dilation. Multiply this velocity by the particle's mass, and you can imagine the particle moving into the future with a certain momentum in its rest frame; observe this particle from a moving frame and its spatial momentum becomes non-zero, and the time component of its momentum (its energy) has to become bigger – the particle packs more punch – as a consequence of the length of the momentum vector being invariant.

A.4.4 Photons

For a photon, the interval represented by $\mathrm{d}\overline{R} \cdot \mathrm{d}\overline{R}$ is always zero ($\mathrm{d}\overline{R} \cdot \mathrm{d}\overline{R} = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 = 0$ for photons. But this means that the proper time $\mathrm{d}\tau^2$ is also zero for photons. This means, in turn, that we cannot define a 4-velocity vector for a photon by the same route that led us to Eq. (A.20), and therefore cannot define a 4-momentum as in Eq. (A.25).

We can do so, however, by a different route. Recall that we defined (in the paragraph following Eq. (A.23)) the 4-velocity as a vector pointing along the world line, which resulted in the 4-momentum being in the same direction. From the discussion of the momentum of massive particles in the previous section, we see that the P^0 component is related to the energy, so we can use this to define a 4-momentum for a *massless* particle, and again write

$$\overline{P}_\gamma = (E, \mathbf{p}_\gamma).$$

Since the photon's velocity 4-vector is null, the photon's 4-momentum must be also (since it is defined in Eq. (A.25) to be pointing in the same direction). Thus we must have $\overline{P}_\gamma \cdot \overline{P}_\gamma = 0$, thus $\mathbf{p}_\gamma \cdot \mathbf{p}_\gamma = E^2$, recovering the $m = 0$ version of Eq. (A.32),

$$E^2 = p^2 \qquad \text{(massless particle)}, \tag{A.33}$$

so that even massless particles have a non-zero momentum.

In quantum mechanics, we learn that the energy associated with a quantum of light – a photon – is $E = hf$, where h is Planck's constant, $h = 6.626 \times 10^{-34}$ Js (or 2.199×10^{-42} kg m in natural units), so that

$$\overline{P} = (hf, hf, 0, 0) \qquad \text{(photon)}. \tag{A.34}$$

[Exercise A.4]

Exercises

Exercise A.1 (§ 1.1.2) Which of these are inertial frames? (i) A motorway bridge, (ii) a stationary car, (iii) a car moving at a straight line at a constant speed, (iv) a car cornering at a constant speed, (v) a stationary lift, (vi) a free-falling lift. (The last one is rather subtle.)

Exercise A.2 (§ 1.2) I have a friend moving past me in a rocket at a relativistic speed, and I observe her watch to be moving slowly with respect to mine (as we will discover later). She examines my watch as I do this: is it moving faster or slower than hers?

Exercise A.3 (§ 1.2.1) Consider the wave equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = 0. \tag{i}$$

Take

$$\frac{\partial}{\partial x} = \frac{\partial x'}{\partial x}\frac{\partial}{\partial x'} + \frac{\partial t'}{\partial x}\frac{\partial}{\partial t'}$$
$$\frac{\partial}{\partial t} = \frac{\partial x'}{\partial t}\frac{\partial}{\partial x'} + \frac{\partial t'}{\partial t}\frac{\partial}{\partial t'}$$

and so on, and using the GT, show that Eq. (i) does *not* transform into the same form under a GT. $[d^+]$

Exercise A.4 (§ 1.4.4) Consider Compton scattering. We can examine the collision between a photon and an electron. Unlike the classical Thomson scattering of light by electrons, Compton scattering is an inherently relativistic and quantum-mechanical effect, treating both the electron and the incoming light as relativistic particles.

The collision is as shown in Figure A.4. An incoming photon strikes a stationary electron and both recoil. The incoming photon has energy $Q_1 = hf_1 = h/\lambda_1$ and the outgoing one $Q_2 = hf_2 = h/\lambda_2$; the outgoing electron has energy E, spatial momentum $\mathbf{p}$, and mass m.

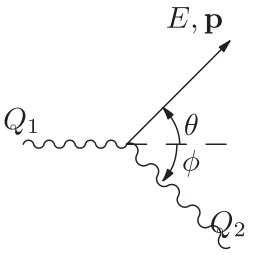

Figure A.4 Compton scattering.

1. Identify the four momentum 4-vectors $\overline{P}_{1e}$, $\overline{P}_{1\gamma}$, $\overline{P}_{2e}$, and $\overline{P}_{2\gamma}$ corresponding to the momenta of the electron and photon before and after the collision.
2. Require momentum conservation, compare components, and show that $\lambda_2 - \lambda_1 = h(1 - \cos\phi)/m$.

Appendix B Solutions to Einstein's Equations

By the end of Chapter 4, we had acquired enough differential geometry to understand where Einstein's equations come from and what they mean, and by extracting Newton's law of gravity from them, in Eq. (4.51), we have now both corroborated Eq. (4.32) and fixed the constant in it. But after bringing you to the threshold of GR, I cannot simply leave you there, so this appendix will briefly describe some solutions to Einstein's equations.

This account is a very brief one; there are fuller versions in most GR textbooks.[1]

B.1 The Schwarzschild Solution

Karl Schwarzschild was the first to obtain an exact solution to Einstein's equations, in December 1915, only a month after Einstein had first published the equations in Berlin[2] (Einstein earlier produced an approximate solution, which obtained the advance in the perihelion of Mercury, and which we will derive below as an encore after the Schwarzschild solution).

Our goal, remember, is to find a set of coordinates, and a metric tensor, which together define an interval

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu \tag{B.1}$$

where the metric satisfies Einstein's equations.

Our first step is to simplify the problem as much as possible, by a sensible choice of coordinates.

[1] This chapter is notationally harmonious with Carroll (2004, chapter 5), which provides very useful expansion of the details. The selection and sequence of ideas, however, follows that in my colleague Martin Hendry's lecture notes, and can thus be claimed as pedagogically verified in the same way as the rest of the book.

[2] Schwarzschild did this work during his service as an army officer in World War I, and described the solution in a letter to Einstein in December 1915, written under the sound of shell-fire, probably in Alsace; Schwarzschild died early the next year, possibly from complications arising from exposure to his own side's experimental chemical weapons. The historical details here are an aside in chapter 12 of Snygg (2012). The book is entertaining in a variety of ways, mathematical and historical, but it is not an introductory text.

The Schwarzschild solution describes a space time that is highly symmetric. It is empty apart from a single mass, which we will take to be at the coordinate system's origin, and it is does not change in time. We can describe the metric using the coordinates (t, r, θ, ϕ): here θ and ϕ are the usual polar coordinates, and although we will take r to be a radial coordinate, we will not assume at this point that it is straightforwardly related to the distance from the origin, nor that t is straightforwardly a proper time.

The time-independence of the metric means two things. Firstly, that none of the components of the metric, $g_{\mu\nu}$ has a dependence on t; and secondly, that the interval includes no time-space cross terms, $\mathrm{d}t\,\mathrm{d}x^i$ (if the metric is time-independent, then it must be invariant under time reversal, but if g_{0i} were non-zero, then a change of time coordinate from t to $-t$ would change the sign of $\mathrm{d}t\,\mathrm{d}x^i$ terms, but no others).

The rotational symmetry of the space-time must be reflected in a rotational symmetry of the metric: it must be invariant under rotations about the origin. What that means is that the (2-d) space at constant r (and constant t) has the geometry of the surface of a sphere, or $\mathrm{d}s^2 = \rho(r)\mathrm{d}\Omega^2 \equiv \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2$ for some function ρ that depends *only* on the coordinate r. The function ρ doesn't depend on t, because none of the coefficients does, and it doesn't depend on θ or ϕ, because if it did then the spaces of constant (t, r) would not be rotationally symmetric. This is what it means to identify r as 'a radial coordinate'. This also means that there are no $\mathrm{d}r\,\mathrm{d}\theta$ or $\mathrm{d}r\,\mathrm{d}\phi$ cross terms.

We have by now excluded a large number of possible terms in the fully general metric, and can now write this (with some foreknowledge) as

$$\mathrm{d}s^2 = -e^{2\alpha(r)}\mathrm{d}t^2 + e^{2\beta(r)}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2. \tag{B.2}$$

Here the choice of signs is to retain compatibility with the Minkowski metric $\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\mathrm{d}\Omega^2$, and we have assumed that the coefficients of $\mathrm{d}t^2$ and $\mathrm{d}r^2$ cannot change sign from this. Also, we have *chosen* the coordinate r to be such that the metric on surfaces of constant (t, r) is $r^2\mathrm{d}\Omega^2$ (and if we hadn't, for some reason, then we could change radial variables $r \mapsto \bar{r}$ so that the coefficient $\rho(\bar{r})$ was indeed just $\bar{r}^2$).

Notice that this illustrates the principle of general covariance, of Section 1.1. Given a space-time – such as the space-time around a single point mass – we are allowed to choose how we wish to label points within it, using a set of four independent coordinates (four, because the space-time we are interested in is homeomorphic to $\mathbb{R}^4$ – see Section 3.1.1). We can explore the length of intervals in that space-time using a clock, for timelike intervals, or a piece of string, for spacelike intervals, and summarise the structure we thus discover, by writing down the coefficients of the metric. Those coefficients depend on our instantaneous position within the space-time, *and* how we have chosen to label them using our coordinates. The covariance principle declares that the physical conclusions we come to must not change with a change in coordinates – for example that a dropped object accelerates radially downwards or that the curl of a magnetic field is proportional to a current; the numerical value of the acceleration might depend on the coordinate choice, or the constant of proportionality, but not the underlying geometrical statements. Similarly, it is the covariant derivative we defined in Section 3.3.2 that allows us to define differentiation in such a way that the derivative depends on the metric but, again, not the choice of coordinates.

The above account may seem somewhat hand-waving, but it is the intuitive content of a more formal account, which depends on identifying the geometrical structures that generate the symmetries contained within a particular manifold. This discussion leads to *Birkhoff's theorem*, which asserts that the Schwarzschild metric, which we are leading up to, is the only spherically symmetric vacuum solution (static or not). See Carroll (2004, §5.2).

With this metric, the Christoffel symbols are:

$$\begin{aligned}
\Gamma^t_{tr} &= \alpha' & \Gamma^r_{tt} &= \mathrm{e}^{2(\alpha-\beta)}\alpha' & \Gamma^r_{rr} &= \beta' \\
\Gamma^\theta_{r\theta} &= 1/r & \Gamma^r_{\theta\theta} &= -r\mathrm{e}^{-2\beta} & \Gamma^\phi_{r\phi} &= 1/r \\
\Gamma^r_{\phi\phi} &= -r\mathrm{e}^{-2\beta}\sin^2\theta & \Gamma^\theta_{\phi\phi} &= -\sin\theta\cos\theta & \Gamma^\phi_{\theta\phi} &= \frac{\cos\theta}{\sin\theta}
\end{aligned} \tag{B.3}$$

(others zero) where primes denote differentiation with respect to r. The corresponding Ricci tensor components are

$$\begin{aligned}
R_{tt} &= \mathrm{e}^{2(\alpha-\beta)}\left[\alpha'' + \alpha'^2 - \alpha'\beta' + 2\alpha'/r\right] \\
R_{rr} &= -\alpha'' - \alpha'^2 + \alpha'\beta' + 2\beta'/r \\
R_{\theta\theta} &= \mathrm{e}^{-2\beta}\left(r(\beta' - \alpha') - 1\right) + 1 \\
R_{\phi\phi} &= \sin^2\theta R_{\theta\theta}
\end{aligned} \tag{B.4}$$

(off-diagonal elements zero).[3]

Now we have the Ricci tensor, we can use the constraint $R_{\mu\nu} = 0$, Eq. (4.30), to obtain the metric.

Since both R_{tt} and R_{rr} are zero, so is their sum, and so

$$0 = \mathrm{e}^{2(\beta-\alpha)}R_{tt} + R_{rr} = \frac{2}{r}(\alpha' + \beta'), \tag{B.5}$$

which implies $\alpha + \beta$ is constant. Since we want the metric, Eq. (B.2), to reduce to the Minkowski metric at large r, we must have both $\alpha(r) \to 0$ and $\beta(r) \to 0$ as $r \to \infty$. Thus the constant is zero, and

$$\alpha = -\beta.$$

With this, we can rewrite $R_{\theta\theta} = 0$ as

$$1 = \mathrm{e}^{2\alpha}(2r\alpha' + 1) = \frac{\partial}{\partial r}(r\mathrm{e}^{2\alpha}),$$

which we can promptly solve to obtain

$$\mathrm{e}^{2\alpha} = 1 - \frac{R}{r}, \tag{B.6}$$

where R is a constant of integration.

To find a value for this constant, R, we can return to the geodesic equation, Eq. (3.43). Consider a test particle released from rest, so with $\mathrm{d}x^i/\mathrm{d}\tau = 0$, as usual taking τ to be the proper time, $\mathrm{d}\tau^2 = -\mathrm{d}s^2$. Looking back at the metric of Eq. (B.2), we see that

[3] Relativists really *like* computer algebra packages.

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \mathrm{e}^{-\alpha},$$

so that the r-component of the geodesic equation is

$$\frac{\mathrm{d}^2 r}{\mathrm{d}\tau^2} + \Gamma^r_{tt}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 = \frac{\mathrm{d}^2 r}{\mathrm{d}\tau^2} + \mathrm{e}^{2\alpha}\alpha' = 0,$$

or, differentiating Eq. (B.6),

$$\frac{\mathrm{d}^2 r}{\mathrm{d}\tau^2} = -\frac{R}{2r^2}. \tag{B.7}$$

In the non-relativistic, or weak field, limit, the acceleration of a test particle due to gravity is (compare Section 4.3.2)

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = -\frac{GM}{r^2},$$

where M is the mass of the body at $r = 0$. From this we can see that R, the *Schwarzschild radius*, is

$$R = 2GM, \tag{B.8}$$

and putting this back into the prototype metric Eq. (B.2), we have

$$\mathrm{d}s^2 = -\left(1 - \frac{2GM}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2GM}{r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 \tag{B.9a}$$

$$\equiv g_{tt}\mathrm{d}t^2 + g_{rr}\mathrm{d}r^2 + g_{\theta\theta}\mathrm{d}\theta^2 + g_{\phi\phi}\mathrm{d}\phi^2, \tag{B.9b}$$

the metric for the Schwarzschild solution to Einstein's equations. We will also refer to (t, r, θ, ϕ) as (x^0, x^1, x^2, x^3) below.

Differentiating $\mathrm{e}^{2\alpha} = \mathrm{e}^{-2\beta} = 1 - R/r$ to obtain

$$\alpha' = -\beta' = \frac{R}{2r}\frac{1}{r - R},$$

we can specialise Eq. (B.3) to

$$\begin{aligned}
\Gamma^t_{tr} &= \frac{R}{2r(r-R)} & \Gamma^r_{tt} &= \frac{R}{2r^3}(r-R) & \Gamma^r_{rr} &= \frac{-R}{2r(r-R)} \\
\Gamma^\theta_{r\theta} &= 1/r & \Gamma^r_{\theta\theta} &= -(r-R) & \Gamma^\phi_{r\phi} &= 1/r \\
\Gamma^r_{\phi\phi} &= -(r-R)\sin^2\theta & \Gamma^\theta_{\phi\phi} &= -\sin\theta\cos\theta & \Gamma^\phi_{\theta\phi} &= \frac{\cos\theta}{\sin\theta}.
\end{aligned} \tag{B.10}$$

The sun has a mass of approximately $2 \times 10^{30}\,\mathrm{kg}$ in physical units. Taking $G = 2/3 \times 10^{-10}\,\mathrm{m^2\,kg^{-1}\,s^{-2}}$, this gives $R = 8/3 \times 10^{20}\,\mathrm{m^3\,s^{-2}}$; if we then divide by the conversion factor $1 = c^2 = 9 \times 10^{16}\,\mathrm{m^2\,s^{-2}}$, we end up with $R \approx 3 \times 10^3\,\mathrm{m}$. Equivalently, we may recall the natural units of Section 4.3, in which the choice of $G = 1$ creates a conversion factor between units of kilogrammes and units of metres, exactly as we did when writing $c = 1$ in Section 1.4.1. In these units, the mass of the sun is approximately $1.5 \times 10^3\,\mathrm{m}$ giving, again, $R = 2M = 3 \times 10^3\,\mathrm{m}$. From here on, we will work in units where $G = 1$.

As you can see from Eq. (B.9), the coefficients have a singularity at $r = 2M$. These coordinates are ill-behaved at $r = 2M$ but, it turns out, there is no physical

singularity there. It is not trivial to demonstrate this, but it is possible to find alternative coordinates – Kruskal–Szekeres coordinates – that are perfectly well-behaved at that point. Although the space-time is not singular at $r = 2M$, that radius is nonetheless special. The radius $r = 2M$ is known as the 'event horizon', and demarcates two causally separate regions: there are no world lines that go from inside the event horizon to the outside, so no events within it can affect the space-time outside.

[Exercise B.1]

B.2 The Perihelion of Mercury

Now that we have a metric for the Schwarzschild space-time – that is, a solution to Einstein's equations – our next step is to ask about the dynamics in that space-time – that is, how free-fall test particles move within it. This matters because the Schwarzschild solution is an adequate model for the space-time round any single gravitating mass. So the question crystallises into: what are the orbits of planets within the solar system, viewed as a Schwarzschild space-time?

First, notice that there are no cross-terms in Eq. (B.9), in these coordinates – that is, $g_{ij} = 0$ for $i \neq j$; these are known as an *orthogonal coordinates*. In the case of orthogonal coordinates, the matrix of metric components g_{ij} is diagonal (which implies that $g^{ii} = 1/g_{ii}$), and the geodesic equation simplifies to

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(g_{\mu\mu}\frac{\partial x^\mu}{\partial \tau}\right) - \frac{1}{2}\sum_\alpha g_{\alpha\alpha,\mu}\left(\frac{\partial x^\alpha}{\partial \tau}\right)^2 = 0 \qquad \text{(no sum on } \mu\text{)}. \tag{B.11}$$

This will simplify some of the calculations below. See Exercise B.2.

We can see that $g_{\mu\nu}$ is independent of both t and ϕ, so that the second term in Eq. (B.11) disappears for both $\mu = 0$ and $\mu = 3$, and thus that the geodesic equations for t and ϕ are solvable as

$$g_{tt}\frac{\mathrm{d}t}{\mathrm{d}\tau} = k \qquad \text{constant} \tag{B.12}$$

$$g_{\phi\phi}\frac{\mathrm{d}\phi}{\mathrm{d}\tau} = h \qquad \text{constant}. \tag{B.13}$$

Similarly, the θ-equation ($\mu = 2$) becomes

$$r^2\frac{\mathrm{d}^2\theta}{\mathrm{d}\tau^2} + 2r\frac{\mathrm{d}r}{\mathrm{d}\tau}\frac{\mathrm{d}\theta}{\mathrm{d}\tau} - r^2\sin\theta\cos\theta\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2 = 0$$

(using intermediate results for the metric, from Exercise B.1), which has $\theta = \pi/2$ as a particular integral (a test particle which starts in that plane stays in that plane).

Trying to do the same thing for r, in Eq. (B.11), gives us a second order differential equation to solve, which we'd prefer to avoid. Instead, we can use the fact that

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(g_{\alpha\beta}\frac{\partial x^\alpha}{\partial \tau}\frac{\partial x^\beta}{\partial \tau}\right) = 0 \tag{B.14}$$

along a geodesic (see Exercise B.3). Integrating this, we can fix the constant of integration by recalling that $U^\mu U_\mu = -1$ in Minkowski space, to give

$$g_{\alpha\beta}\frac{\partial x^\alpha}{\partial\tau}\frac{\partial x^\beta}{\partial\tau} = -1,$$

and thus, on inserting the coefficients of the metric,

$$-1 = g_{tt}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 + g_{rr}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + g_{\phi\phi}\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2,$$

or

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 = k^2 - 1 - \frac{h^2}{r^2} + \frac{2M}{r}\left(1 + \frac{h^2}{r^2}\right). \tag{B.15}$$

The next step is to change variables from r to $u \equiv 1/r$, and to regard ϕ rather than τ as the independent variable, so that $\mathrm{d}r/\mathrm{d}\phi = \mathrm{d}r/\mathrm{d}\tau\,\mathrm{d}\tau/\mathrm{d}\phi$. This mirrors the conventional approach to the solution of the Kepler problem; both Carroll (2004) and Schutz (2009) touch on this, and it is discussed in detail in Goldstein (2001, chapter 3). Using this change of variables, we find

$$h^2\left(\frac{\mathrm{d}u}{\mathrm{d}\phi}\right)^2 = -(1 - Ru)(1 + h^2u^2) + k^2.$$

If we now differentiate this, we can rearrange the result to obtain

$$\frac{\mathrm{d}^2u}{\mathrm{d}\phi^2} = 3Mu^2 - u + \frac{M}{h^2} \tag{B.16}$$

(switching to $M = R/2$, here, avoids a few unsightly powers of 2 below). If you trace this calculation back, you see that the first term, in u^2, is associated with the factor of $1 - R/r$ in $\mathrm{d}t/\mathrm{d}\tau$, Eq. (B.12); it is, loosely speaking, the relativistic term, and it is the term that is *not* present in the corresponding part of the non-relativistic, or newtonian, calculation. And sure enough, this term is much smaller than the other two terms, for cases such as the earth's orbit around the sun.

What that means in turn is that the solution to this equation will be the solution to the Kepler problem, u_0, plus a small correction, which we can write as

$$u = u_0 + u_1 \qquad (u_1 \ll u_0).$$

You can confirm that, in these units, the solution to the newtonian problem (that is, Eq. (B.16) without the relativistic term) is

$$u_0(\phi) = \frac{M}{h^2}(1 + e\cos\phi),$$

and thus

$$\frac{\mathrm{d}^2u_1}{\mathrm{d}\phi^2} + u_1 = 3\frac{M^3}{h^4}\left(1 + \frac{e^2}{2} + 2e\cos\phi + \frac{e^2}{2}\cos 2\phi\right), \tag{B.17}$$

where we have suppressed terms in $u_0 u_1$ and u_1^2, and recalled that $\cos^2\phi = (1 + \cos 2\phi)/2$. If you stared at this for long enough, you would doubtless spot that

$$\frac{\mathrm{d}^2}{\mathrm{d}\phi^2}(\phi \sin\phi) + \phi \sin\phi = 2\cos\phi$$
$$\frac{\mathrm{d}^2}{\mathrm{d}\phi^2}(\cos 2\phi) + \cos 2\phi = -3\cos 2\phi,$$

which prompts us to write

$$u_1(\phi) = A + \frac{B}{2}\phi \sin\phi - \frac{C}{3}\cos 2\phi, \tag{B.18}$$

which, on substutition into Eq. (B.17), gives

$$A + B\cos\phi + C\cos 2\phi = 3\frac{M^3}{h^4}\left(1 + \frac{e^2}{2} + 2e\cos\phi + \frac{e^2}{2}\cos 2\phi\right),$$

giving values for A, B, and C, which are all numerically small.

Examining the terms in Eq. (B.18), we see that the first term is a (small) constant, and the third oscillates between $\pm C/2$; both, therefore, have negligible effects. The middle term, however, has an oscillatory factor, but also a secular factor, which grows linearly with ϕ.

If we finally add $u_0 + u_1$ again, but discard the negligible terms in A and C, then we obtain

$$u = \frac{M}{h^2}(1 + e\cos(1 - \alpha)\phi), \qquad \alpha = \frac{3M^2}{h^2} \ll 1 \tag{B.19}$$

(using a Taylor expansion which shows that $\cos(1 - \alpha)\phi = \cos\phi + \alpha\phi\sin\phi + O(\alpha^2)$). This is very nearly an ellipse, but with a perihelion that advances by an angle $2\pi\alpha$ per orbit.

We can describe a newtonian orbital ellipse with

$$u(\phi) = \frac{1 + e\cos\phi}{a(1 - e^2)},$$

where e is the orbital eccentricity, and a the semi-major axis. By comparison with the non-relativistic part of Eq. (B.16), we find that

$$\frac{M}{h^2} = \frac{1}{a(1 - e^2)},$$

and thus that

$$\Delta\phi \equiv 2\pi\alpha = 6\pi\frac{M^2}{h^2} = \frac{6\pi M}{a(1 - e^2)}.$$

The nearest planet to the sun is Mercury. The actual orbit of Mercury is not quite a kelperian ellipse, but instead precesses at 574 arcsec/century (relative to the ICRF, the relevant inertial frame). This is almost all explicable by newtonian perturbations arising from the presence of the other planets in the solar system, and over the course of the nineteenth century much of this anomalous precession had been accounted for in detail. The process of identifying the corresponding perturbations had also been carried out

for Uranus, with the anomalies in that case resolved by predicting the existence of, and then finding, Neptune.[4] At one point it was suspected that there was an additional planet near Mercury which could explain the anomaly, but this was ruled out on other grounds, and a residual precession of 43 arcsec/century remained stubbornly unexplained.

Mercury has semi-major axis $a = 5.79 \times 10^{10}\,\mathrm{m} = 193\,\mathrm{s}$, and eccentricity $e = 0.2056$. Taking the sun's mass to be $M_\odot = 1.48\,\mathrm{km} = 4.93 \times 10^{-5}\,\mathrm{s}$, we find $\Delta\phi = 5.03 \times 10^{-7}\,\mathrm{rad\,orbit}^{-1}$. The orbital period of Mercury is 88 days, so that converting $\Delta\phi$ for Mercury to units of arcseconds per century, we find

$$\Delta\phi = 43.0\,\mathrm{arcsec/century}.$$

Einstein published this calculation in 1916. It is the first of the classic tests of GR, which also include the deflection of light (or other EM radiation) in its passage near the limb of the sun, and the measurement of gravitational redshift. Numerous further tests have been made, at increasing precision, and no deviations from GR's predictions have been found. The history of such tests is discussed at some length in Will (2014).

[Exercise B.2–B.4]

B.3 Gravitational Waves

The third application we will look at, in this appendix, is that of gravitational waves – how they propagate and how they are detected. This will be a rather compact account, which will mostly follow the sequence of ideas in Carroll (2004, chapter 7), because it makes it easy to pick up the ideas of gauge invariance that we met in passing in Section 4.3.1. Schutz (2009, chapter 9) gets to the same endpoint more directly. See those references (or indeed many others) for the details I have elided below.

B.3.1 Einstein's Equations in the Transverse-Traceless Gauge

Our starting point is the 'nearly-Minkowskian' metric of Eq. (4.36), where the matrix $h_{\alpha\beta}$ is a symmetric matrix of small components. We write the components of this matrix as follows:

$$\begin{aligned} h_{00} &= -2\Phi \\ h_{0i} &= w_i \\ h_{ij} &= 2s_{ij} - 2\Psi\delta_{ij}, \end{aligned} \tag{B.20}$$

where s_{ij} is traceless, so that the trace of $h_{\alpha\beta}$ appears only in $\Psi = -\delta^{ij}h_{ij}/6$. The tensor s_{ij} is referred to as the *strain*. We have done nothing here other than change the notation; in particular, note that there are ten degrees of freedom here, just as in the original h_{ij}. The significance of the different partitions of the matrix is that each of the 00, 0i, and ij components transforms into itself under a spatial rotation.

[4] This took place in a few years leading up to 1846, and led to a Franco-British priority dispute over which country's astronomer had made and published the crucial prediction (Kollerstrom 2006).

From this metric it is straightforward but tedious[5] to obtain first the Christoffel symbols, and then the Riemann tensor

$$\begin{aligned} R_{0j0l} &= \Phi_{,lj} + w_{(l,j0)} - h_{jl,00} \\ R_{0jkl} &= w_{[l,k]j} - h_{j[l,k]0} \\ R_{ijkl} &= h_{i[l,k]j} - h_{j[l,k]i} \end{aligned} \tag{B.21}$$

and Einstein tensor

$$\begin{aligned} G_{00} &= 2\nabla^2\Psi + s^{ij}{}_{,ij} \\ G_{0i} &= -\tfrac{1}{2}\nabla^2 w_j + \tfrac{1}{2}w^k{}_{,kj} + 2\Psi_{,j0} + s_j{}^k{}_{,k0} \\ G_{ij} &= \left(\delta_{ij}\nabla^2 - \frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}\right)(\Phi - \Psi) + \delta_{ij}w^k{}_{,k0} - w_{(j,i)0} \\ &\quad + 2\delta_{ij}\Psi_{,00} - \Box s_{ij} + 2s^k{}_{(j,i)k} - \delta_{ij}s^{kl}{}_{,kl}. \end{aligned} \tag{B.22}$$

Recall that the Roman indexes run over only the spatial indexes; that the $\Box$ symbol is the *d'Alembertian* operator,

$$\Box \equiv \eta^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu} = -\frac{\partial^2}{\partial t^2} + \nabla^2,$$

the four-dimensional version of the Laplacian; the definition of the symmetrisation notation in Eq. (4.42); and that these expressions are calculated using $h_{\mu\nu} \ll \eta_{\mu\nu}$.

Having quoted the Einstein tensor, our next task is to find a way of throwing most of it away.

The Einstein tensor is governed by Einstein's equation, $G_{\mu\nu} = 8\pi T_{\mu\nu}$. Examining the G_{00} term, we can see that the corresponding term in Einstein's equation specifies Ψ in terms of s_{ij} and T_{00}. There is no time derivative of Ψ, and so no propagation of it. Similarly, the G_{0i} term specifies the w_j in terms of Ψ, s_{ij} and the T_{0i}, and the G_{ij} term specifies Φ, in each case, again, without a time derivative. So although, after Eq. (B.20), we counted ten degrees of freedom, they are not all independent. The propagating terms – the terms that will shortly lead to a wave equation – are all in the s_{ij} component of the metric, which, you may recall, is both symmetric and trace-free.

In the discussion under Eq. (4.41) we discussed the family of gauge transformations generated by $h_{\alpha\beta} \to h_{\alpha\beta} + 2\xi_{(\alpha,\beta)}$ (this is mildly adjusted from Eq. (4.40), to make the signs below nicer). What does this look like, when applied to the metric parameterised by Eq. (B.20)? We obtain

$$\begin{aligned} \Phi &\to \Phi + \xi^0{}_{,0} \\ w_i &\to w_i + \xi^i{}_{,0} - \xi^0{}_{,i} \\ \Psi &\to \Psi - \frac{1}{3}\xi^k{}_{,k} \\ s_{ij} &\to s_{ij} + \xi_{(j,i)} - \frac{1}{3}\xi^k{}_{,k}\delta_{ij}. \end{aligned} \tag{B.23}$$

[5] Did I mention that relativists *really* like computer algebra packages?

Under this transformation,

$$s^{ik}{}_{,k} \to s^{ik}{}_{,k} + \xi^{(k,i)}{}_{,k} - \frac{1}{3}(\xi^{k}{}_{,k}\delta^{ij})_{,j}$$
$$= s^{ik}{}_{,k} + \frac{1}{6}\xi^{k,i}{}_{,k} + \frac{1}{2}\nabla^2\xi^{i}.$$

This gives a differential equation for ξ^i which makes the divergence of the strain vanish

$$s^{ik}{}_{,k} = 0. \tag{B.24}$$

Doing the same thing with the w^i expression, and choosing ξ^0 so that $\nabla^2\xi^0 - \xi^i{}_{,0i} = w^i{}_{,i}$ gives

$$w^i{}_{,i} = 0.$$

This choice of coordinates – that is, this choice of ξ^μ when applied to x^μ – is known as the *transverse gauge*. This gauge choice significantly simplifies Einstein's equations in Eq. (B.22). In this gauge, we can finally solve Einstein's equations to find propagating gravitational wave solutions.

We are looking for solutions propagating in free space, in which $T_{\mu\nu} = 0$. Looking at the simplified Eq. (B.22), the $G_{00} = 0$ equation implies $\nabla^2\Psi = 0$, which, in an asymptotically flat space-time (remember $\Psi = -h_{00}/2 \ll \eta_{00}$), implies $\Psi = 0$. The G_{0i} term then similarly implies $w_i = 0$. Finally, $G_{ij} = 0$ implies that Tr $G_{ij} = \eta^{ij}G_{ij} = 0$ and thus (since s_{ij} is traceless) that $\nabla^2\Phi = 0$ and $\Phi = 0$. The only term that survives this carnage is the traceless part of the $G_{ij} = 0$ equation, or

$$\Box s_{ij} = 0. \tag{B.25}$$

This is usually referred to as the *transverse-traceless gauge*. Looking back to our re-notation of the metric perturbation $h_{\mu\nu}$ in Eq. (B.20), and recalling that in this context Φ, w_i, and Ψ are all zero, we write the metric perturbation in this gauge as h^{TT}, and rewrite the above reduction of Einstein's equations as

$$\Box h^{\mathrm{TT}}_{\mu\nu} = 0. \tag{B.26}$$

Comparing this to the definitions of Φ, w_i, and Ψ in Eq. (B.20), we can see that $h^{\mathrm{TT}}_{00} = h^{\mathrm{TT}}_{0i} = 0$. Eq. (B.26) has as solution a matrix of metric perturbations $h^{\mathrm{TT}}(x^\alpha) = h^{\mathrm{TT}}(t, \mathbf{x})$, giving the small deviations from a Minkowski metric as a function of position and time. Recall that this matrix gives expressions for the components of this perturbation *in this gauge*, and that (recalling Section 4.3.1) we can regard this as a tensor in a background Minkowski space.

B.3.2 Gravitational Wave Solutions

Equation (B.26) is a wave equation, which has solutions

$$h^{\mathrm{TT}}_{\mu\nu} = C_{\mu\nu}\exp \mathrm{i}k_\alpha x^\alpha(\lambda), \tag{B.27}$$

for a constant one-form k_α, and a constant tensor C with possibly complex components. In this gauge, $C_{00} = C_{0i} = 0$ and $\eta^{\mu\nu}C_{\mu\nu} = 0$ (i.e., C is traceless).

Applying Eq. (B.26) to this, we find that

$$-k_\alpha k^\alpha h^{\mathrm{TT}}_{\mu\nu} = 0,$$

which is a solution only if $k_\alpha k^\alpha = 0$, so that the wave-vector k^α is null.

Looking back at the solution in Eq. (B.27), we see that $h^{\mathrm{TT}}_{\mu\nu}$ is constant if $k_\alpha x^\alpha$ is constant. This is true for the curve $x^\alpha(\lambda) = k^\alpha f(\lambda) + l^\alpha$ for any scalar function f and constant vector l^α. Here, the function $f(\lambda) = \lambda$, or some affine transformation of that, gives the worldline of a photon, indicating that a given perturbation – that is, a given value of $h^{\mathrm{TT}}_{\mu\nu}$ – propagates in the same way as a photon, rectilinearly at the speed of light.

Imposing the gauge condition ${h^{\mathrm{TT}\mu\nu}}_{,\nu} = 0$, Eq. (B.24), we deduce that $k_\mu C^{\mu\nu} = 0$: the wave vector is orthogonal to the constant tensor $C_{\mu\nu}$, and the oscillations of the solution are transverse to the direction of propagation. This is why this is called the transverse gauge.

Our last observation is that we can, without loss of generality, orient our coordinates so that the wave vector is pointing along the z-axis, and $k^\mu = (\omega, 0, 0, \omega)$, writing ω for the time-component of the wave vector. Thus $0 = k^\mu C_{\mu\nu} = k^0 C_{0\nu} + k^3 C_{3\nu}$ so that $C_{3\nu} = 0$ as well. Writing (with, again, a little foreknowledge) $C_{11} = h_+$ and $C_{12} = h_\times$, this means that the tensor components $C_{\mu\nu}$ simplify, in this gauge, to

$$C_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{B.28}$$

After identifying ten degrees of freedom in Eq. (B.20), in this gauge there are only two degrees of freedom left.

B.3.3 Detecting Gravitational Waves

Above, in Eqs. (B.27) and (B.28), we have identified *gravitational waves*. What is the effect of these on matter, or, in more pragmatic terms, how can we detect them experimentally? First, we need to ask what is the effect, on a free-falling particle, of a passing gravitational wave.

The paragraph before Eq. (B.28) describes a set of coordinate systems where h^{TT} is a perturbation of a Lorentz frame oriented so that the gravitational wave is propagating along the z-axis. Given a test particle in free fall, there is one of these frames with respect to which the particle is initially at rest, with tangent vector $U^\mu = (1, 0, 0, 0)$. This particle obeys the geodesic equation, Eq. (3.43),

$$\frac{\mathrm{d}}{\mathrm{d}\tau} U^\mu + \Gamma^\mu_{\alpha\beta} U^\alpha U^\beta = 0.$$

Given the initial condition, this gives an initial deflection of

$$\frac{\mathrm{d}}{\mathrm{d}\tau} U^\mu = -\Gamma^\mu_{00} = -\frac{1}{2}\eta^{\mu\nu}(h^{\mathrm{TT}}_{0\nu,0} + h^{\mathrm{TT}}_{\nu 0,0} - h_{00,\nu}).$$

Since h^{TT} is proportional to $C_{\mu\nu}$, and $C_{\mu\nu}$ has components as in Eq. (B.28) in these coordinates, we see that the right-hand side is zero, so that the particle's velocity vector is unchanged, and it *remains at rest* as the gravitational wave passes.

Consider now a second test particle, a distance ϵ from the first, along the x-axis. The proper distance between these two particles is

$$\begin{aligned}\Delta l = \int |\mathrm{d}s^2|^{1/2} &= \int |g_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta|^{1/2} \\ &= \int_0^\epsilon |g_{xx}|^{1/2}\mathrm{d}x \\ &\approx |g_{xx}|^{1/2}\Big|_{x=0}\epsilon \\ &= \left(1 + \tfrac{1}{2}h^{\mathrm{TT}}_{xx}\Big|_{x=0} + O\left(h^{\mathrm{TT}2}_{xx}\right)\right)\epsilon. \end{aligned} \tag{B.29}$$

Since h^{TT} has an oscillatory factor, the proper distance between the test particles oscillates as the wave passes, even though both particles are following geodesics, and are thus 'at rest' in their corresponding Lorentz frames. They are 'at rest' in the *separate* senses that they remain at the same coordinate positions (in these coordinates) *and* that they remain on geodesics. This is an example of the distinction between 'acceleration' as a second derivative of coordinate position – which is a frame-dependent thing – and acceleration as the thing you directly perceive when pushed – which is a frame-independent thing.

The two test particles are both following geodesics, so we should be able to analyse this same situation using the geodesic deviation equation, Eq. (3.60), which directly describes the way in which the proper distance between two test particles changes, as they move along their mutual geodesics. To first order, we can take both tangent vectors to be $U^\mu = (1,0,0,0)$, as above, so that

$$\left(\frac{\mathrm{d}^2\overline{\xi}}{\mathrm{d}t^2}\right)^\mu = R^\mu{}_{\alpha\beta\nu}U^\alpha U^\beta \overline{\xi}^\nu.$$

Evaluating $R^\mu{}_{\alpha\beta\nu} = \eta^{\mu\sigma}R_{\sigma\alpha\beta\nu}$ in our transverse-traceless gauge,

$$\begin{aligned} R_{\sigma\alpha\beta\nu}U^\alpha U^\beta &= h^{\mathrm{TT}}_{\sigma[\nu,0]0} - h^{\mathrm{TT}}_{0[\nu,0]\sigma} && \text{from Eq. (B.21)} \\ &= \tfrac{1}{2}\left(h^{\mathrm{TT}}_{\sigma\nu,00} - h^{\mathrm{TT}}_{\sigma 0,\nu 0} - h^{\mathrm{TT}}_{0\nu,0\sigma} + h^{\mathrm{TT}}_{00,\nu\sigma}\right) \\ &= \tfrac{1}{2}h^{\mathrm{TT}}_{\sigma\nu,00} && \text{from Eq. (B.28).} \end{aligned}$$

For these test particles, $x^0 = \tau = t$, so that the geodesic deviation equation becomes

$$\left(\frac{\mathrm{d}^2\overline{\xi}}{\mathrm{d}t^2}\right)^\mu = \tfrac{1}{2}\eta^{\mu\sigma}\xi^\nu\frac{\mathrm{d}^2}{\mathrm{d}t^2}h^{\mathrm{TT}}_{\sigma\nu}. \tag{B.30}$$

Notice that only ξ^1 and ξ^2 are affected – the directions transverse to the direction the wave is travelling in.

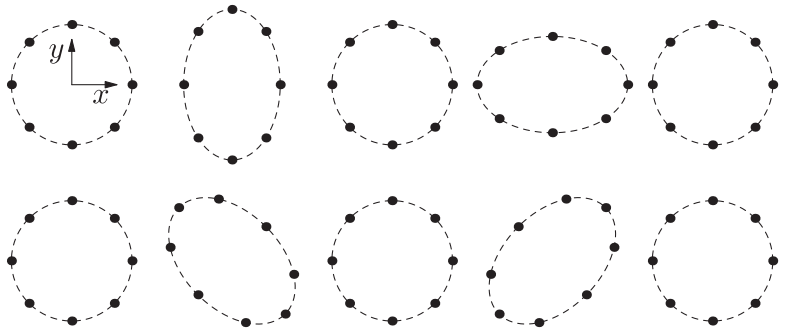

Figure B.1 Polarised gravitational waves. The successive figures show the positions of a ring of test particles at successive times. The upper figure shows the effect of + -polarised gravitational waves, and the lower the effect of × -polarised ones.

If $h_\times = 0$, then Eq. (B.30) becomes

$$\begin{aligned}\frac{\partial^2 \xi^1}{\partial t^2} &= \tfrac{1}{2}\xi^1 \frac{\partial^2}{\partial t^2} h_+ \exp \mathrm{i} k_\mu x^\mu \\ \frac{\partial^2 \xi^2}{\partial t^2} &= -\tfrac{1}{2}\xi^2 \frac{\partial^2}{\partial t^2} h_+ \exp \mathrm{i} k_\mu x^\mu,\end{aligned}$$

with lowest-order solutions

$$\begin{aligned}\xi^1 &= \left(1 + \tfrac{1}{2} h_+ \exp \mathrm{i} k_\mu x^\mu\right) \xi^1(0) \\ \xi^2 &= \left(1 - \tfrac{1}{2} h_+ \exp \mathrm{i} k_\mu x^\mu\right) \xi^2(0).\end{aligned} \tag{B.31}$$

The corresponding solutions for $h_+ = 0$ are

$$\begin{aligned}\xi^1 &= \xi^1(0) + \tfrac{1}{2} h_\times \exp(\mathrm{i} k_\mu x^\mu) \xi^2(0) \\ \xi^2 &= \xi^2(0) + \tfrac{1}{2} h_\times \exp(\mathrm{i} k_\mu x^\mu) \xi^1(0).\end{aligned} \tag{B.32}$$

Compare Eq. (B.29): what we have done here is essentially a more elaborate calculation of the same thing.

The effects of these solutions, on a ring of test particles, are shown in Figure B.1, with the + -polarised solutions corresponding to Eq. (B.31) and the × -polarised ones corresponding to Eq. (B.32). Since Eq. (B.30) is linear, its solutions are superpositions of these two solutions.

The gravitational waves emitted by moving masses – in practical terms, by fast-orbiting or collapsing stellar masses – propagate at the speed of light in these two polarisations.

The orbit of the two neutron stars in the so-called 'Hulse–Taylor' binary pulsar has slowed in exactly the way that GR would predict, if the binary system is radiating gravitational waves. This provides a very strong *indirect* detection of gravitational waves Weisberg et al. (2010). However the calculations above suggest that these waves in space-time can also be directly detected by measuring the proper separations of test particles in that space-time.

If a pair of test masses – either two of the particles in Figure B.1 or the two in Eq. (B.29) – are not in free fall but instead carefully suspended so that they are stationary in a lab, then their motion is govered both by the suspending force, which is accelerating them away from a free-fall geodesic, but which is constant, and by the changes to space-time caused by any passing gravitational waves. By carefully monitoring the changes in proper separation between the masses, using the light-travel time between them, we should be able to detect the passage of any gravitational waves in the vicinity.

The size of these fluctuations is given by the size of the terms in h^{TT}, compared to the terms in the metric they are perturbing, which are equal to 1. For the Hulse–Taylor binary mentioned above, $h^{\mathrm{TT}} \sim 10^{-23}$. Detecting the changes in proper separation requires a rather intricate measurement.

One way of doing this is to develop resonant bar detectors, which are large bars, typically aluminium, with a mechanical resonance that could be excited by a sufficiently strong and sufficiently long gravitational wave of the matching frequency. These proved insufficiently sensitive, however, and the gravitational wave community instead moved on to develop interferometric detectors. These use an optical arrangement very similar in principle to the Michelson–Morley detector which failed to detect the luminiferous aether, and which was so important in the history of SR. The test masses here are mirrors at the ends of two arms at right angles to each other: changes in the lengths of the arms, as measured by lasers within the arms are, with suitably heroic efforts, detectable interferometrically.

There are detailed discussions of the calculation in Schutz (2009, chapter 9) and Carroll (2004, chapter 7), a summary calculation and a discussion of astrophysical implications in Sathyaprakash and Schutz (2009), and an account of the experimental design in Pitkin et al. (2011). The LISA design for a future space-based interferometer, at a much larger scale and with higher sensitivities, and the astronomy it may unlock, is described in Amaro-Seoane et al. (2013 & 2017). Pitkin et al. (2011) also provides a very brief history of the development of gravitational wave detectors, and there is a much more extensive history, from the point of view of the sociology of science, in Collins (2004).

The first direct detection of gravitational waves was made by the LIGO experiment on 2015 September 14, and jointly announced by the LIGO and Virgo collaborations on 2016 February 11 (Abbott et al., 2016). Subsequent detections confirm the expectation that such measurements will become routine, and will allow gravitational wave physics to change from being an end in itself, to becoming a non-electromagnetic messenger for the exploration of the universe on the largest scales.

Exercises

Exercise B.1 (§ 2.1) Derive Eqs. (B.3) and (B.4). You might as well obtain Eq. (B.10) as an encore. [d^-]

Exercise B.2 (§ 2.2) Prove Eq. (B.11). Suspend the summation convention for this exercise. First, obtain the Christoffel symbols for an orthogonal metric:

$$\begin{aligned}
\Gamma^{\alpha}_{\alpha\alpha} &= \frac{g_{\alpha\alpha,\alpha}}{2g_{\alpha\alpha}} \\
\Gamma^{\alpha}_{\beta\beta} &= \frac{-g_{\beta\beta,\alpha}}{2g_{\alpha\alpha}} && \alpha \neq \beta \\
\Gamma^{\alpha}_{\alpha\beta} &= \frac{g_{\alpha\alpha,\beta}}{2g_{\alpha\alpha}} && \alpha \neq \beta \\
\Gamma^{\alpha}_{\beta\gamma} &= 0 && \alpha, \beta, \gamma \text{ all different.}
\end{aligned}$$

Then start from one or other version of the geodesic equation, such as Eq. (4.46), expand $\mathrm{d}/\mathrm{d}\tau(g_{\mu\mu}\,\partial x^{\mu}/\partial\tau)$, use the geodesic equation, and expand the sum carefully. [u^{+}]

Exercise B.3 (§ 2.2) Prove Eq. (B.14). Expand the left-hand side, use the geodesic equation, and use symmetry.

Exercise B.4 (§ 2.2) Estimate the relative numerical values of the terms on the right-hand side of Eq. (B.16), for the orbit of the earth. Confirm that the first term is indeed small compared to the third. [u^{+}]

Appendix C Notation

Chapters 2 and 3 introduce a great deal of sometimes confusing notation. The best way to get used to this is: to get used to it, by working through problems, but while you're slogging there, this crib might be useful.

C.1 Tensors

A $\binom{M}{N}$ tensor is a linear function of M one-forms and N vectors, which turns them into a number. A $\binom{1}{0}$ tensor is called a vector, and a $\binom{0}{1}$ tensor is a one-form. Vectors are written with a bar over them, $\overline{V}$, and one-forms with a tilde $\widetilde{p}$ (§2.2.1). In my (informal) notation in the text, $\mathsf{T}(\widetilde{\cdot}, \overline{\cdot})$ is a $\binom{1}{1}$ tensor – a machine with a single one-form-shaped slot and a single vector-shaped slot. Note that this is a different beast from $\mathsf{T}(\overline{\cdot}, \widetilde{\cdot})$, which is also a $\binom{1}{1}$ tensor, but with the slots differently arranged.

C.2 Coordinates and Components

In a space of dimension n, a set of n linearly independent vectors $\overline{e}_i$ ($i = 1, \ldots, n$) forms a basis for all vectors in the space; a set of n linearly independent one-forms $\widetilde{\omega}^i$ forms a basis for all one-forms in the space.

$\overline{e}_i(\widetilde{\omega}^j) = \delta_i{}^j \Leftrightarrow \widetilde{\omega}^j(\overline{e}_i) = \delta^j{}_i.$	*Choose* basis vectors and one-forms to be *dual* (remember that $\overline{e}_i$ and $\widetilde{\omega}^i$ are functions)	(2.5)
$\overline{V} = V^0\overline{e}_0 + V^1\overline{e}_1 + \cdots$	Vectors have components, written with raised indexes	§2.2.5
$\widetilde{p} = p_0\widetilde{\omega}^0 + p_1\widetilde{\omega}^1 + \cdots$	. . . so do one-forms, but written with lowered indexes	
$V^i = \overline{V}(\widetilde{\omega}^i), \quad p_i = \widetilde{p}(\overline{e}_i)$	Components of vectors and one-forms (a consequence of the preceding relations)	(2.6)

$T^i{}_j = \mathsf{T}(\widetilde{\omega}^i, \overline{e}_j)$	Tensors have components, too	(2.7)
$T_i{}^j = \mathsf{T}(\overline{e}_i, \widetilde{\omega}^j)$	A different beast (note the arrangement of indexes)	

The object $T^i{}_j$ is a *number* – a component of a tensor in a particular basis. However we also (loosely, wickedly) use this same notation to refer to the corresponding *matrix* of numbers, and even to the corresponding $\binom{1}{1}$ tensor T.

The vector space in which these objects live is the tangent plane to the manifold M at the point P, $T_P(M)$ (§3.1.1). In this space, the basis vectors are $\overline{e}_i = \partial/\partial x^i$ (§3.1.2), and the basis one-forms $\widetilde{\mathrm{d}}x^i$, where x^i is the i-th coordinate (more precisely, coordinate function; note that x^i is *not* a component of any vector, though the notation makes it look like one). These bases are dual: $\overline{e}_i(\widetilde{\omega}^j) = \partial/\partial x^i(\widetilde{\mathrm{d}}x^j) = \delta_i{}^j$ (cf., (3.7)).

C.3 Contractions

A contraction is a tensor with some or all of its arguments filled in.

$\overline{V}(\widetilde{p}) = \widetilde{p}(\overline{V})$	by choice	§2.2.5
$\widetilde{p}(\overline{V}) \equiv \langle \widetilde{p}, \overline{V} \rangle$	special notation for vectors contracted with one-forms	(2.2)
$\widetilde{p}(\overline{V}) = p_i V^i$	basis independent	§2.2.5
$\mathsf{T}(\widetilde{p}, \widetilde{\cdot}, \overline{V})^j = p_i V^k T^{ij}{}_k$	partially contracted $\binom{2}{1}$ tensor (a vector)	§2.2.5
$\overline{p} = \mathsf{g}(\widetilde{p}, \widetilde{\cdot})$	a vector, with components…	§3.2.3
$\mathsf{g}(\widetilde{p}, \widetilde{\omega}^j) = p_i \mathsf{g}(\widetilde{\omega}^i, \widetilde{\omega}^j) \equiv p^j$	definition of vector $\overline{p}$, with raised indexes, dual to one-form $\widetilde{p}$, written with lowered indexes	
$g_{ij} = \mathsf{g}(\overline{e}_i, \overline{e}_j),$	components of the metric	(2.12)
$g^{ij} = \mathsf{g}(\widetilde{\omega}^i, \widetilde{\omega}^j)$	…up and down	
$g_{ij} g^{kl} T^j{}_l = T_i{}^k$	the metric raises and lowers indexes; $\mathsf{T}(\widetilde{\cdot}, \overline{\cdot})$ and $\mathsf{T}(\overline{\cdot}, \widetilde{\cdot})$ are distinct but related	
$g_{ij}, \quad g^i{}_j = \delta^i{}_j, \quad g^{ij}$	different tensors in principle, but all referred to as ‘the metric’	(2.15)

C.4 Differentiation

$V^i{}_{,j} \equiv \partial V^i/\partial x^j$	(non-covariant) derivative of a vector component	

$p_{i;j} \equiv \partial p_i / \partial x^j$	... and one-form	
$\nabla \overline{V}$	covariant derivative of $\overline{V}$ (a $\binom{1}{1}$ tensor)	§3.2.2
$\nabla_{\overline{U}} \overline{V}$	covariant derivative of $\overline{V}$ in the direction $\overline{U}$ (a vector)	cf. (3.40)
$\nabla_{\overline{e}_i} \overline{V} \equiv \nabla_i \overline{V}$	shorthand	
$(\nabla \overline{V})^i{}_j = V^i{}_{;j}$	components of the $\binom{1}{1}$ tensor $\nabla \overline{V}$	(3.21)
$(\nabla \widetilde{p})_{ij} = p_{i;j}$	components of the $\binom{0}{2}$ tensor $\nabla \widetilde{p}$	
$V^{i;j} = g^{jk} V^i{}_{;k}$	it's a tensor, so you can raise its indexes, too	
$\nabla_{\overline{U}} \overline{V} = U^\alpha V^\beta{}_{;\alpha} \overline{e}_\beta$	... putting all that together	

C.5 Changing Bases

We might change from a basis $\overline{e}_i$ (for example $\overline{e}_0, \ldots, \overline{e}_3$) to a basis $\overline{e}_{\bar{\imath}}$ ($\overline{e}_{\bar{0}}, \ldots, \overline{e}_{\bar{3}}$), noting that the bar goes over the index and not, as might be more intuitive, the vector (that is, we don't write $\overline{\overline{e}}_i$ or $\overline{e}'_i$). The transformation is described using a matrix (it's not a tensor) $\Lambda^{\bar{\imath}}_{j}$ (§2.2.7).

$\overline{e}_{\bar{\imath}} = \Lambda^i_{\bar{\imath}} \overline{e}_i, \quad \widetilde{\omega}^{\bar{\imath}} = \Lambda^{\bar{\imath}}_i \widetilde{\omega}^i$	transformation of basis vectors and one-forms	§3.1.4
$V^{\bar{\imath}} = \Lambda^{\bar{\imath}}_i V^i, \quad p_{\bar{\imath}} = \Lambda^i_{\bar{\imath}} p_i$	transformation of components	(2.17)
$\overline{e}_i = \Lambda^{\bar{\jmath}}_i e_{\bar{\jmath}} = \Lambda^{\bar{\jmath}}_i \Lambda^j_{\bar{\jmath}} \overline{e}_j$	the transformation matrix goes in both directions	
$\Rightarrow \Lambda^{\bar{\jmath}}_i \Lambda^j_{\bar{\jmath}} = \delta^j_i$	matrix inverses	(2.22)

Notes: (1) these look complicated to remember, but as long as each Λ has one barred and one unbarred index, you'll find there's only one place to put each index, consistent with the summation convention. (2) This shows why it's useful to have the bars on the indexes. (3) Some books use hats for the alternate bases: $\overline{e}_{\hat{\imath}}$.

C.6 Einstein's Summation Convention

See §2.2.5.

$$p_i V^i \equiv \sum_i p_i V^i.$$

The convention works only for *two* repeated indexes: one up, one down. This is one of the reasons why one-form components are written with lowered indexes and vectors

with raised ones; the other is to distinguish the components p_i of the one-form $\widetilde{p}$ from the components $p^i = g^{ij}p_j$ of the related vector $\overline{p} = \mathsf{g}(\widetilde{p}, \widetilde{\cdot})$. Points to watch:

- A term should have at most two duplicate indexes: one up, one down; if you find something like U^iV^i or U^iT_{ii}, you've made a mistake.
- All the terms in an expression should have the same unmatched index(es): $A^i = B^jT^i{}_j + C^i$ is all right, $A^i = B^jT^k{}_j$ is a mistake (typo or thinko).
- You can change or swap repeated indexes: $A^{ij}T_{ij}$, $A^{ik}T_{ik}$, and $A^{ji}T_{ji}$ all mean exactly the same thing, but all are different from $A^{ij}T_{ji}$ (unless T happens to be symmetric).

However, sometimes we will refer to particular components of a tensor or matrix, such as referring to the diagonal elements of the metric as g^{ii} – there's no summation convention here, so the proscriptions above aren't relevant. The context matters.

C.7 Miscellaneous

$\delta^i{}_j = \delta^{ij} = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{else} \end{cases}$	Kronecker delta symbol	§2.1.1
$\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$	the metric (tensor) of Minkowski space	(2.33)
$[\overline{A}, \overline{B}] = \overline{A}\,\overline{B} - \overline{B}\,\overline{A}$	the 'commutator'	before (3.57)
$A_{[ij]} = (A_{ij} - A_{ji})/2$	anti-symmetrisation	
$A_{(ij)} = (A_{ij} + A_{ji})/2$	symmetrisation	(4.42)

For the definitions – and specifically the sign conventions – of the Riemann, Ricci, and Einstein tensors, see Eqs. (3.49), (4.16), and (4.23). For a table of contrasting sign conventions in different texts, see Table 1.1 in §1.4.3.

Note: $\Lambda^i_{\bar{j}}$ and Γ^i_{jk} are matrices, not the components of tensors, so the indexes don't correspond to arguments, and so don't have to be staggered.

In general, component indexes are Roman letters: i, j, and so on. When discussing specifically space-time, it is traditional but not universal to use Greek letters such as μ, ν, α, β, and so on, for component indexes ranging over $\{0, 1, 2, 3\}$, and Roman letters for indexes ranging over the spacelike components $\{1, 2, 3\}$. Here, we use the latter convention only in Chapter 4.

Components are usually standing in for numbers, however we'll sometimes replace them with letters when a particular coordinate system suggests them. For example $\overline{e}_x$ rather than $\overline{e}_1$ in the context of cartesian coordinates, or write $\Gamma^\theta_{\phi\phi}$ rather than, say, Γ^1_{22} when writing the Christoffel symbols for coordinates (θ, ϕ). There shouldn't be confusion (context, again), because x, y, θ, and ϕ are never used as (variable) component indexes; see e.g., Eq. (3.15).

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